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ABSTRACT

This guide represents the final experimental version of a pilot project which was conducted in the United States between 1973 and 1976. The ideas and the manner of presentation are based on the works of Georges and Frederique Papy. They are recognized for having introduced colored arrow drawings ("papygrams") and models of our numeration system (the Papy "minicomputer") into the teaching of mathematics at the elementary and secondary level in Belgium. This program follows the "spiral approach." This guide includes the following topics: (1) multiplication; (2) multiples and divisors; (3) rational numbers; (4) abacuses and algorithmic thinking; (5) standard algorithms of arithmetic; (6) composition; and (7) hand-calculator puzzles. Old and new concepts are covered within these topics. The situations provided are intended to develop problem-solving skills and an understanding of mathematical ideas and concepts in the world of numbers. (Author/SA)

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CSMP Mathematics for the Intermediate Grades

Part IV

Teacher's Guide

U.S. DEPARTMENT OF HEALTH,
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NATIONAL INSTITUTE OF
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The World of Numbers

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Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades

Part IV

The World of Numbers

CEMREL, Inc.

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THE WORLD OF NUMBERS

INTRODUCTION

By this time, your students have had a rich variety of experiences with various kinds of numbers and operations and relations on them. These include encountering both positive and negative integers, decimal numbers, some rational numbers, numerical functions (e.g., 2x; +3; ÷10; −5; $\frac{3}{4}$x), order relations (e.g., < and >), and the notions of multiples and positive divisors of a given number. They have been introduced to paper and pencil algorithms for addition, subtraction, and multiplication of whole numbers and decimal numbers, and they have had extensive experiences using systematic methods for division and for multiplication and addition of rational numbers (fractions) in preparation for algorithms.

In the present strand, these previous numerical experiences will be revisited, extended, and deepened. New situations will be presented, including the introduction of several fascinating extensions and modifications of numerical games: Guess My Rule (Section 1.3, Section 2.3, and Section 2.6); Binary Abacus (Section 4.2 and Section 4.4); Newtonian Quiz (Section 3.2); and Detective Stories (Chapter 2); and new numerical games: Nabu's Abacus (Section 4.1 and Section 4.3) and Hand-Calculator Puzzles (Chapter 7). Old and new concepts will be explored in the context of these situations.

The first paragraph above, describing some of your students' previous numerical experiences, simply catalogues some of the realms of the world of numbers your students have visited. It does not indicate the travel routes nor the vehicles used to travel them.

The CSMP curriculum stresses the unity and continuity of the growth of mathematical ideas and concepts. This is reflected in numerous ways in this strand, as in the others. For example, CSMP employs the spiral approach, not pausing for mastery of each lesson but rather allowing students to encounter the elements of each content strand in different situations, interspersed with other strands throughout the year. It is important to recognize this approach consciously. If you strive for mastery of each single lesson, you will find yourself involved in a great deal of redundancy as the year progresses.

Further, CSMP presents the content in a situational framework. That is, a "pedagogy of situations" is employed which engages students in rich problem solving activities. These offer the opportunity both to develop necessary numerical skills and to gain deeper understanding of mathematical ideas and concepts in the world of numbers. At the same time, the situations presented encourage students to develop patterns of logical thinking and strategies for attacking problems.

Perhaps the most important embodiments of the CSMP approach are the three nonverbal languages used throughout its elementary program: the Papy Minicomputer, the language of arrows, and the language of strings. These are the vehicles which make it possible to develop the situations in which the content is presented to students. Further, as students grow in the CSMP curriculum, these languages become basic investigatory tools available to explore the content.

It is hard to overstate the value of developing tools which aid in attacking problems as well as in representing situations; tools which are not confined to one area of mathematical content or to one level of the development of content. In the usual elementary curriculum, students must continually learn new languages to express each new piece of mathematics that is studied. Armed with the universally applicable languages of the CSMP curriculum, the students grow more and more familiar with the syntax of these languages and are freed to explore new content directly in terms of these same languages rather than being forced to think of each new mathematical idea as tied to a certain new language. This is definitely not to say that the CSMP students do not learn the usual descriptive language of mathematics. However, in the CSMP approach the usual descriptive language is not the vehicle for learning, but only for describing the ideas.

For specific examples of the situations and the ways CSMP uses nonverbal languages in this strand, we refer you to the short introductory remarks preceding each chapter and, in particular, to the lessons themselves.

A WORD ABOUT STANDARD ALGORITHMS

We mentioned above that CSMP seeks to develop the necessary skills in the numerical domain as well as deeper understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that along with the growth of understanding in the world of numbers, there must be a concomitant growth of familiarity and facility with numbers and the operations and relations on them. But facility must not be confused with understanding. They are partners in the growth of mathematical maturity, and it is extremely important that neither be sacrificed for the other. Rather, a balanced growth in each must be maintained.

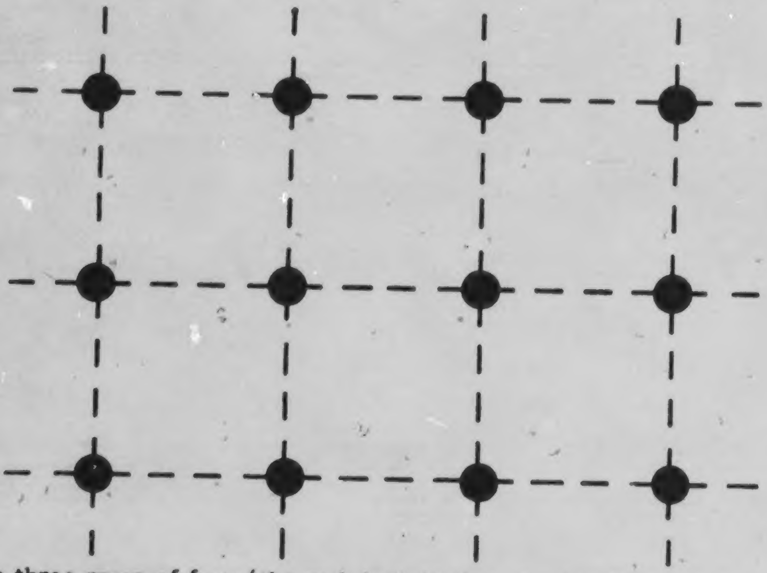
In the usual elementary curriculum, paper and pencil algorithms for the four basic operations provide a prime opportunity for the abuse of the position stated above. It is undeniably important that a student eventually learns mechanical algorithms for the basic operations. However, if these algorithms are presented prematurely they stunt the student's ability to develop alternative algorithms, to do mental arithmetic, to estimate, and so on.

For example, rather than employing the usual division algorithm to determine the standard name for 1,791 ÷ 9, it may be easier and more efficient to note that 200 x 9 = 1,800, so that 199 x 9 = 1,791 and 1,791 ÷ 9 = 199. Indeed, built into this way of approaching the problem is a relatively good estimate (1,800 ÷ 9) of the number involved.

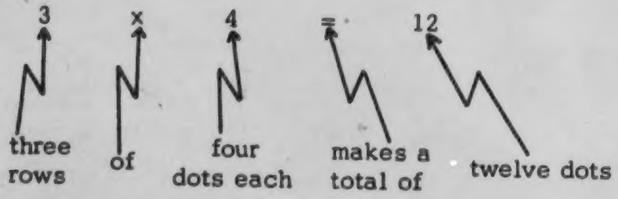
If we insist on a mechanistic response to problems such as the one above (i.e., insist on the use of paper and pencil algorithms), we may not only encourage inefficiency in many situations but also inhibit the development of flexibility necessary for problem solving. On the other hand, if we present to students a rich array of situations in which they interact with numbers and their operations and let the standard algorithms grow naturally as just one facet of the students' experience, we obtain two important results. The students will gain the necessary facility with standard algorithmic procedures and at the same time retain the openness required to respond creatively to new situations in the world of numbers.

NOTE ON THE ORDER OF MULTIPLICATION

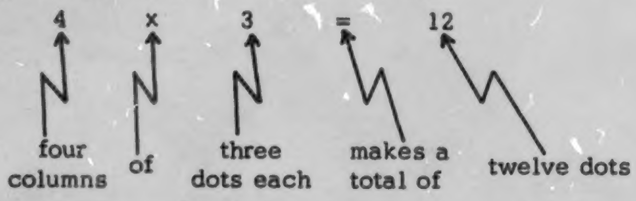
Should the function "multiply by 2" by denoted by "2 ×" or "× 2"? Does it matter? The answer to this last question is "yes and no". It depends on your viewpoint and on the context in which you want to use it. In order to appreciate this fact a little better, let us go back to the very beginning: multiplication of nonnegative whole numbers. How does the following array of 12 dots appear to you?



If you see three rows of four (the red dashed lines), this fact can be recorded as:



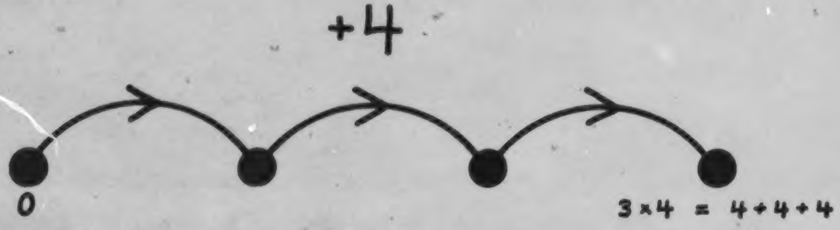
On the other hand, if you see four columns of three (the blue dashed lines), this fact can be recorded as:



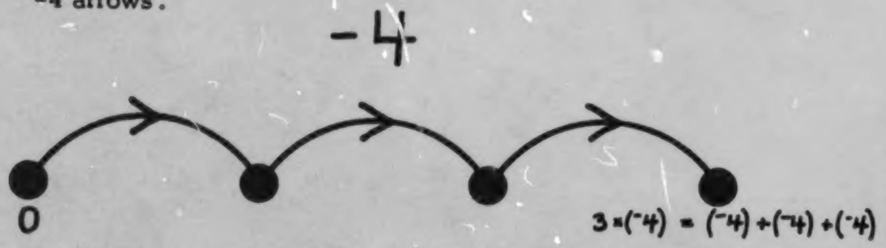
In that they describe two different perceptions, "4 × 3" and "3 × 4" are obviously very different. But, equally obvious is that "4 × 3" and "3 × 4" are both names for the same number, namely 12. So, in this case, it makes no difference if we denote "multiply by 3" by "×3" or "3×". This is true for any nonnegative integer n: "×n" and "n×" may be used indifferently to denote "multiply by n", provided the number you are multiplying by n is also a nonnegative integer.

Problems do not arise until negative integers and (non-integer) decimal numbers put in their appearance; the reason being that it is a bit more difficult to find a model for multiplication when negative integers and (non-integer) decimal numbers are around. What would it mean, for example, to have an array of dots with six rows each containing negative two dots, or, even worse, negative three rows each containing negative five dots? What about 3.72 rows each containing seven dots? Fortunately, it is possible to find a model which allows the calculation of at least some products involving negative integers or (non-integer) decimal numbers; but, unfortunately, we lose the advantage of the array-of-dots model which makes it evident that the order of multiplication does not influence the answer. We call upon the language of arrows.

- 3 × 4 is the number obtained if you start from 0 and use three +4 arrows.



- 3 × ($^-4$) is the number obtained if you start from 0 and use three -4 arrows.



This model has the advantage that it not only "pictures" the product, but it also provides a means of calculating that product: just label all the dots until you reach the one in which you are interested. The disadvantage is that it doesn't make any sense to speak of a negative number of arrows starting from 0. Thus, strictly speaking, your students have only been introduced to products in which the first number is a whole number (the second number can be either positive or negative or zero and can even be a (non-integer) decimal number). This has implications for notation.

- As noted above, if only nonnegative integers are involved, you can use "2 x" or "x 2" as the fancy takes you (we have scattered both of these notations throughout this semester's curriculum).

- If a negative integer (say, $^-4$) is being multiplied by 2 (say), then, strictly speaking, you should denote this "2 × ($^-4$)" and not "$^-4$ × 2". This is because "2 × ($^-4$)" carries the sense of "($^-4$) + ($^-4$)", whereas "$^-4$ × 2" has no natural connotation at this stage. In other words, strictly speaking, you should use "2 x" when negative integers and (non-integer) decimal numbers are involved, not "x 2". We have observed this rule in the Teacher's Guide for this semester.

We have emphasized the phrase "strictly speaking" above because this is not something you should make into a big issue with your class. In fact, it would be best to avoid all mention of this subject in your classroom if at all possible. If any of your students denote "times" functions one way when strictly speaking they should do it the other way, simply accept it without comment. There could be a variety of reasons why students might do this; among them are:

(i) to conform to a pattern: all the other types of functions are usually written with the operation symbol on the left (−2; ÷3; +6).

(ii) to generalize the results from the case of nonnegative integers: it is perfectly natural to believe that the order of multiplication makes no difference in the case of any integers.

Our concern in this note has been to make you aware of some of the underpinnings of this issue so that, in the very unlikely event that a problem arises in your class, you will have the necessary background information to understand why it might have occurred, thus perhaps pointing the way to a means of getting out of the difficulty.

Of course, you really have nothing to worry about; if your students infringe on any of the above "rules", what they do will not be wrong (because, after all, it really doesn't make any difference in which order you multiply two integers). It is just that, at this stage, what they do will be unfounded.

A WORD ON THE USE OF HAND-CALCULATORS*

The hand-calculator provides another powerful tool for the investigation of problems and situations in the world of numbers. Appropriately, this tool is a natural extension of the Papy Minicomputer, since its internal structure is an electronic simulation of the Minicomputer. This analogy need not be consciously acknowledged by you or by a student, but it does serve to indicate that the hand-calculator is indeed a natural tool in the CSMP curriculum.

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* See General Introduction, Section 2(f) for a more detailed discussion on hand-calculators.

Hand-calculators are recommended materials for Part IV of the Intermediate Grades Program. They are not included in your complete classroom set; however, they may be purchased from CSMP as separate items. There are extensive opportunities for the use of hand-calculators in the lessons of Part IV, especially in Chapters 2, 3, and 7 of this strand. In fact, some lessons require each student to have access to a hand-calculator and would be practically impossible to teach otherwise. Many geometry lessons suggest that hand-calculators provide support for calculations and thereby focus attention on the methods of solving certain problems. The workbooks present, by way of review, situations involving the hand-calculator and can be understood only from previous hands-on experience with calculators. In addition, the supplementary material for the slow learner, "Math Play Therapy", relies heavily on the use of hand-calculators.

We are firmly convinced through our own experience with children that the hand-calculator is a potent pedagogical tool when used properly. The time is not far off when its presence in the classroom will be as commonplace as the pencil. For the present, CSMP will continue to investigate the use of hand-calculators in the classroom, particularly with slow learners, and will develop situations designed for its use.

1

MULTIPLICATION

This chapter provides continued and extended experiences in multiplication with arrow diagrams and the Minicomputer as the principal vehicles. Activities with regular (positive), negative, and weighted checkers on the Minicomputer reinforce mental arithmetic skills with basic multiplication facts. Arrow diagrams support composition of multiplication and division functions; for example, 15 × is 3 × followed by 5 × and $\frac{3}{4}$ × is 3 × followed by ÷ 4. The generalized relation ■ ×, where ■ is any whole number, in arrow pictures stimulates mental arithmetic involving composition of multiplication functions. Patterns in various sequences of numbers obtained from multiplication calculations provide support for properties of multiplication.

CAPSULE LESSON SUMMARIES

Section 1.1:	Decode configurations on the Minicomputer. From a list of numbers, select those that can be put on the Minicomputer with exactly one of the checkers (^), ②, ..., ⑨. Put some numbers on the Minicomputer using a ⑩-checker or a ㊿-checker and one of the above checkers. Change a number on the Minicomputer by a specified amount moving exactly one checker. (Worksheets N01, N02, N03, N04)
Section 1.2:	Label dots in arrow pictures where the arrows are for times some whole number. After labeling a dot, identify the arrow starting at some given number and ending at this dot (or starting at this dot and ending at some given number). Find where other arrows of the relation ■ × can be drawn and identify them using composites; for example, 3 × followed by 15 × is 45 ×. (Worksheets N05, N06, N07, N08)

Section 1.3: Play a Guess My Rule game with the operation * where a * b = (3 × a) + b. Use this rule for * to do some calculations involving *. Play another Guess My Rule game in which the rule is given by arrows for the function 2 × followed by − 5. Label the dots in some arrow roads with arrows for this function.
(Worksheets N09, N10, N11)

Section 1.4: Progressively build an arrow picture with various multiplication and division functions. Locate places where arrows for other multiplication functions (composites of arrows already in the picture) could be drawn. Perform some multiplication calculations using addition and subtraction and results of other related calculations. Observe some patterns in a certain sequence of calculations.
(Worksheets N12, N13, N14)

Section 1.5: Develop short cut methods for adding certain sequences of whole numbers. Observe that the sum of the first n odd whole numbers is a square number, n × n. (Optional) Find an explanation for this result.

1.1 Minicomputer Review

CAPSULE LESSON SUMMARY

Decode configurations on the Minicomputer. From a list of numbers, select those that can be put on the Minicomputer with exactly one of the checkers : (^), ②, . . . , ⑨ . Put some numbers on the Minicomputer using a ⑩-checker or a ㊿-checker and one of the above checkers. Change a number on the Minicomputer by a specified amount moving exactly one checker.

MATERIALS

Teacher: Minicomputer set[*] ; weighted checkers

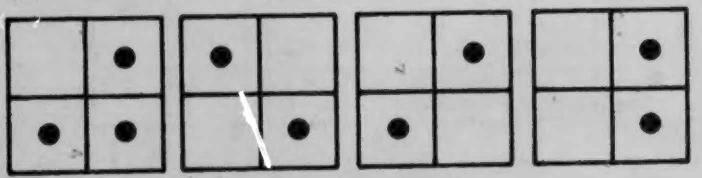
Student: Hand-calculator (optional); paper; pencil; Worksheets N01, N02, N03, N04

DESCRIPTION OF LESSON

Exercise 1

Display four Minicomputer boards and distribute hand-calculators. If you prefer, the decoding of numbers on the Minicomputer can be performed with paper and pencil rather than with hand-calculators.

Slowly put this configuration on the Minicomputer.



T: Put this number on the display of your hand-calculator.

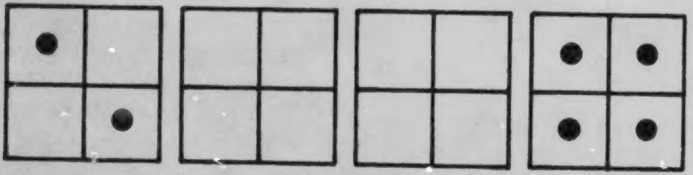
S: 7,965.

---

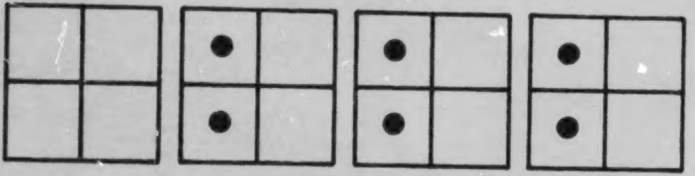
[*] A teacher's Minicomputer set consists of four demonstration Minicomputer boards and a sufficient number of demonstration Minicomputer checkers.

N 1

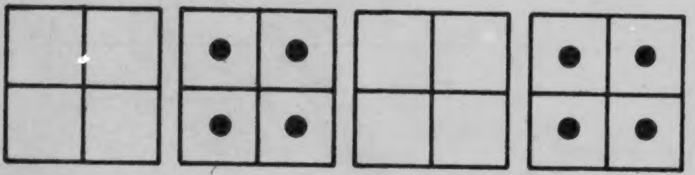
Put these numbers on the Minicomputer and ask the students to display them on their hand-calculators.



= 9,015

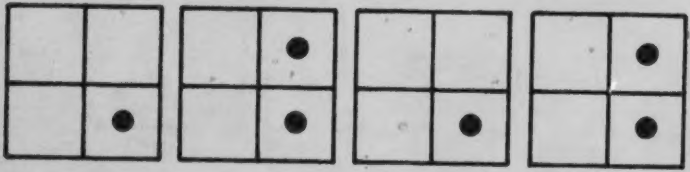


= 1,110



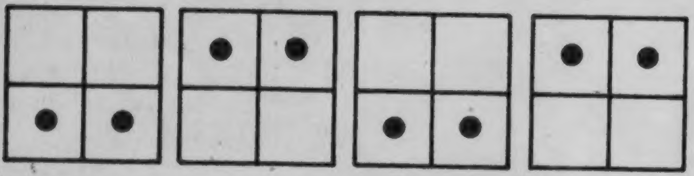
= 1,515

If there are some students who have difficulty decoding this last nonstandard configuration, invite students to make plays to obtain:



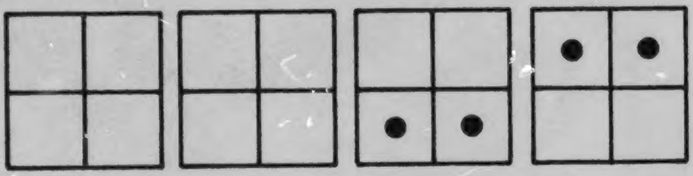
= 1,515

Continue the decoding with this configuration.



= 4,242

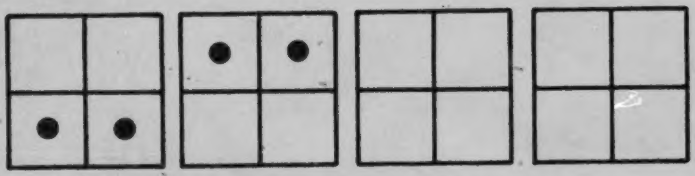
For those students who have trouble decoding this configuration, temporarily remove the checkers from the hundreds' and thousands' boards.



T: What number is this?

S: 42.

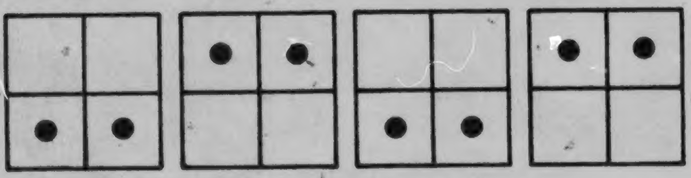
Next, display this configuration.



T: What number is this?

S: 4,200.

Then, return to the original configuration.



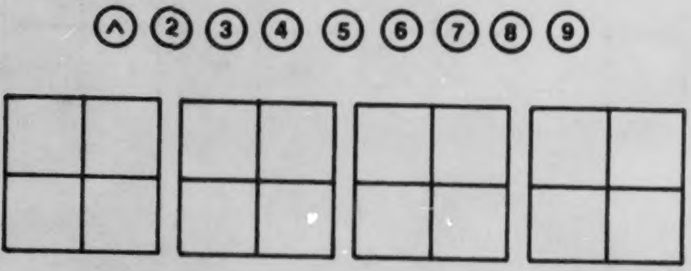
T: What number is this?

S: 4,242.

N 1

Exercise 2

Display four Minicomputer boards and one each of these checkers :



NOTE : Have a ⑩-checker and a ㊿-checker available for use later in this exercise.

Write this list of numbers on the board.

18, 28, 48, 54, 56, 96, 240, 280, 720, 980

T : From this list, find all the numbers that can be put on the Minicomputer using exactly one of these checkers.

Invite students to put numbers on the Minicomputer using exactly one of the checkers. Each time, ask the rest of the class to check that the number on the Minicomputer is indeed one of the numbers in the list.

These seven numbers : 18, 28, 48, 56, 240, 280, and 720 can be put on the Minicomputer using exactly one of the checkers that have been displayed. The configurations on the next page show one way (there may be others) for each number.

N 1

								= 18
						⑨		

							⑦	= 28

						⑥		= 48

						⑦		= 56

					⑥			= 240

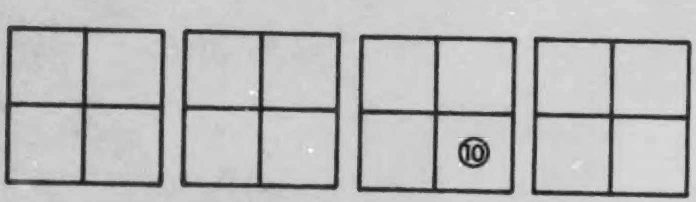
					⑦			= 280

				⑨				= 720

N 1

T: Can you put 96 on the Minicomputer using a ⑩- checker and one other checker?

Ask a student to place just one checker; for example, the student might choose to place a ⑩-checker.



T: What number is on the Minicomputer?

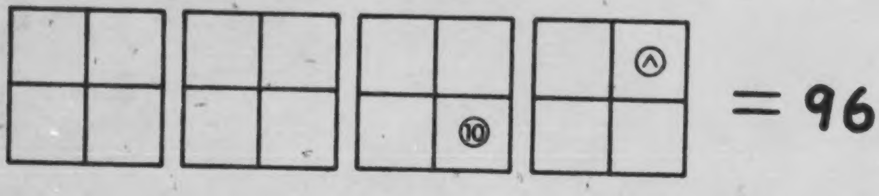
S: 100, because 10 × 10 = 100.

T: How much smaller is 96 than 100?

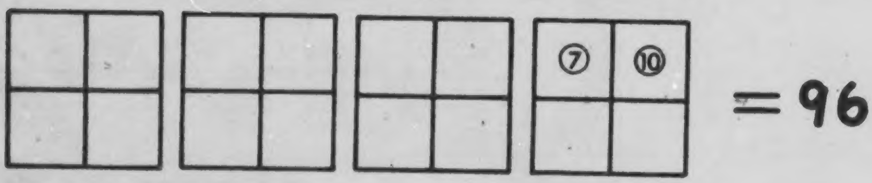
S: 4 smaller.

T: Who can place the second checker?

Invite a student to place the second checker.

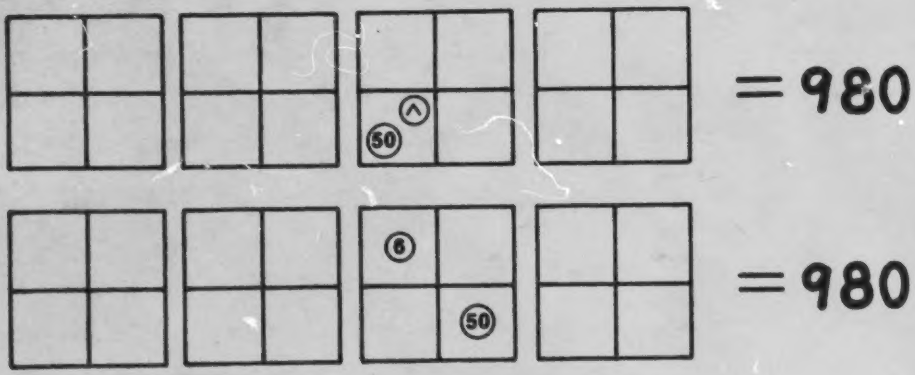


Another solution to this problem is:



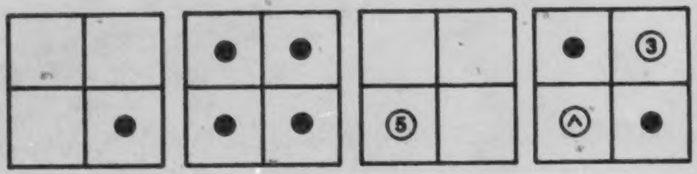
Repeat the above activity with this problem.

Put 980 on the Minicomputer using a (50)-checker and one other checker.



Exercise 3

Display four Minicomputer boards with this configuration of checkers.



T: Look carefully at the number I have put on the Minicomputer. Let's estimate this number. Is this number larger or smaller than 2,000?

S: Larger. There is 1,000 on the thousands' board and 1,000 on the hundreds' board, and there are still more checkers.

T: Is the number larger or smaller than 3,000?

S: Smaller. There is only 2,600 on the thousands', hundreds', and tens' boards together and the number on the ones' board is less than 400.

N1

T: Very good, the number is between 2,600 and 3,000. By moving exactly one checker, can you make the number 90 bigger?

S: Move a checker from the 10-square to the 100-square.

Make the move as the student suggests, then return the checker to its original position and continue the activity by asking similar questions. Feel free to adjust the level of difficulty of these questions to the numerical abilities in your class.

T: Who can make the number:

98 smaller? (Move a positive checker from the 100-square to the 2-square or a negative checker from the 2-square to the 100-square.)

300 bigger? (Move a positive checker from the 100-square to the 400-square or a ⑤-checker from the 20-square to the 80-square.)

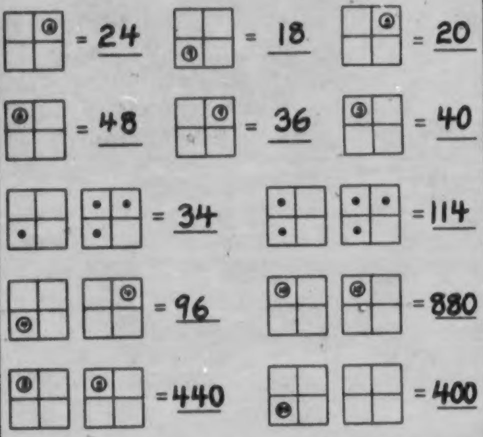
48 bigger? (Move a ③-checker from the 4-square to the 20-square.)

3,992 bigger? (Move a positive checker from the 8-square to the 4,000-square.)

Worksheets N01, N02, N03, and N04 are available.

Name ______________ **N 01 ***

Which number is on the Minicomputer?



= 24 = 18 = 20

= 48 = 36 = 40

= 34 = 114

= 96 = 880

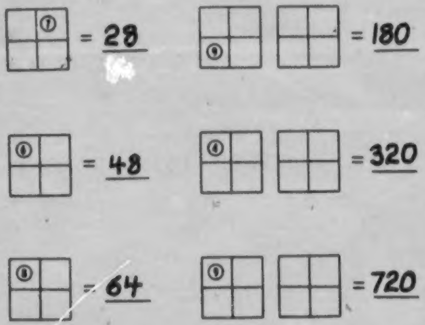
= 440 = 400

Name ______________ **N 02 ****

28 48 54 64 68 180 320 380 720

Put six numbers from this list on the Minicomputer using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



= 28 = 180

= 48 = 320

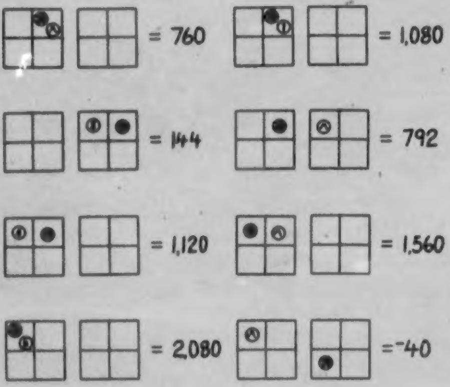
= 64 = 720

(Other solutions are possible.)

Name ______________ **N 03 *****

Put each number on the Minicomputer using a ⑳-checker and exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ^



= 760 = 1,080

= 144 = 792

= 1,120 = 1,560

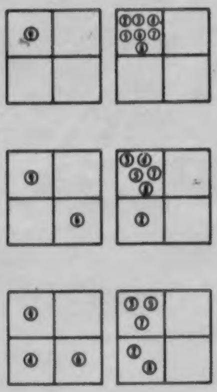
= 2,080 = ⁻40

(Other solutions are possible.)

Name ______________ **N 04 ******

Put 1,000 on the Minicomputer using all of these checkers, each of them exactly once. Try to find at least three solutions.

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



= 1,000

= 1,000

= 1,000

(Many other solutions are possible.)

N 2

1.2 Multiplication #1

CAPSULE LESSON SUMMARY

Label dots in arrow pictures where the arrows are for times some whole number. After labeling a dot, identify the arrow starting at some given number and ending at this dot (or starting at this dot and ending at some given number). Find where other arrows of the relation ■ x can be drawn and identify them using composites; for example, 3 x followed by 15 x is 45 x.

MATERIALS

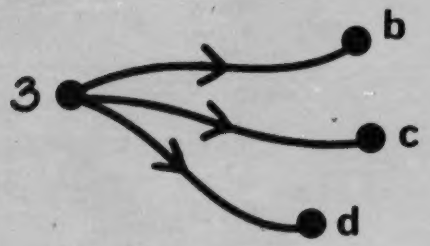
Teacher: Colored chalk
Student: Paper; colored pencils; Worksheets N05, N06, N07, N08

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.

■ x



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: The red arrows are for times some whole number. What number could be here (point to b)?

The class discussion might develop along these lines.

S: 24.

Label b "24".

T: If 24 is here, then we can find a whole number so that this red arrow (trace the arrow from 3 to 24) could be for times that number. What could this red arrow be for?

S: 8×.

Label the appropriate arrow "8×".

Continue in this manner for c and d. Each time ask a student to label a dot, and then find how the corresponding arrow could be labeled. Erase all the labels for dots and arrows except "3". Repeat the activity, with more specific requests for numbers.

T (pointing to b): Give me a number between 40 and 50 that could be here.

S: 45.

T: In this case, what could the red arrow be for?

S: 15×, because 15 × 3 = 45.

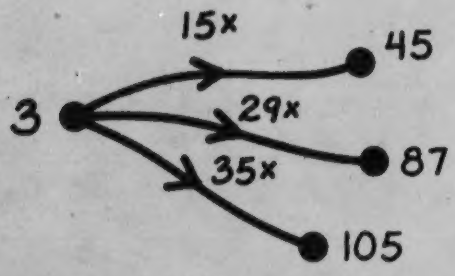
Be sure to label both the dot and the arrow. Continue by requesting for a number:

between 80 and 90 for c. [Answers: 81 (27 × 3 = 81)
or 84 (28 × 3 = 84)
or 87 (29 × 3 = 87)]

between 100 and 110 for d. [Answers: 102 (34 × 3 = 102)
or 105 (35 × 3 = 105)
or 108 (36 × 3 = 108)]

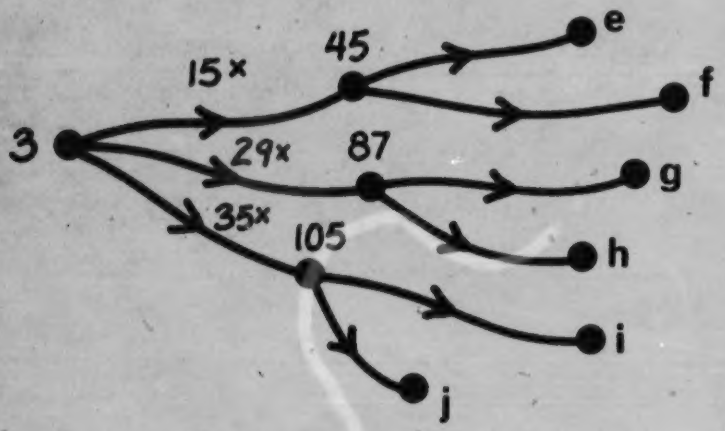
N 2

A completely labeled arrow picture might look like this.



■ x

Extend your arrow picture to the right by drawing six more arrows as shown below.



■ x

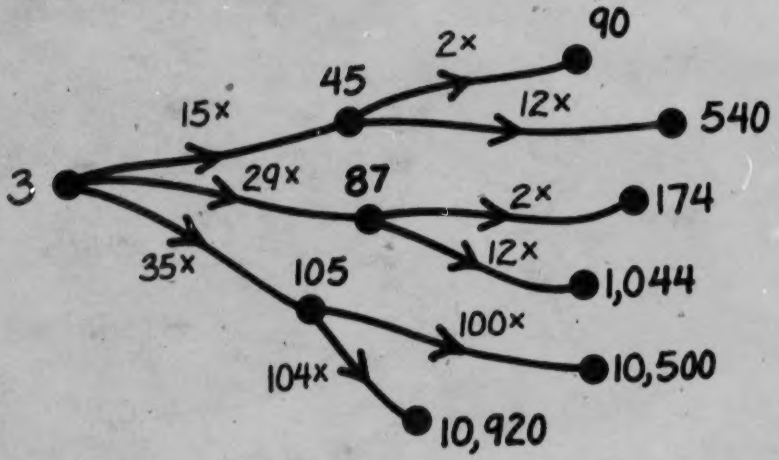
Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

Continue in the same way to label the dots and arrows. Let the students choose any numbers they wish for e and g, but make these more specific requests of numbers for f, h, i, and j. Feel free to adjust the difficulty of your request to the numerical level of your class.

f: a number between 500 and 600.
h: a number between 1,000 and 1,100.
i: a number larger than 10,000.
j: the largest number smaller than 11,000.

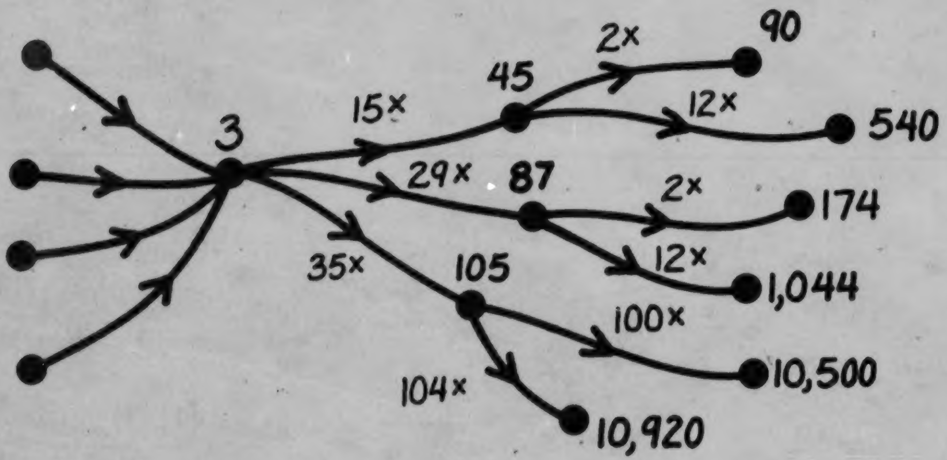
Your completely labeled arrow picture might look like this.

■ ×


3
15×
45
2×
90
12×
540
29×
87
2×
174
12×
1,044
35×
105
100×
10,500
104×
10,920

Add four more red arrows ending at 3 to your picture.

■ ×


3
15×
45
2×
90
12×
540
29×
87
2×
174
12×
1,044
35×
105
100×
10,500
104×
10,920

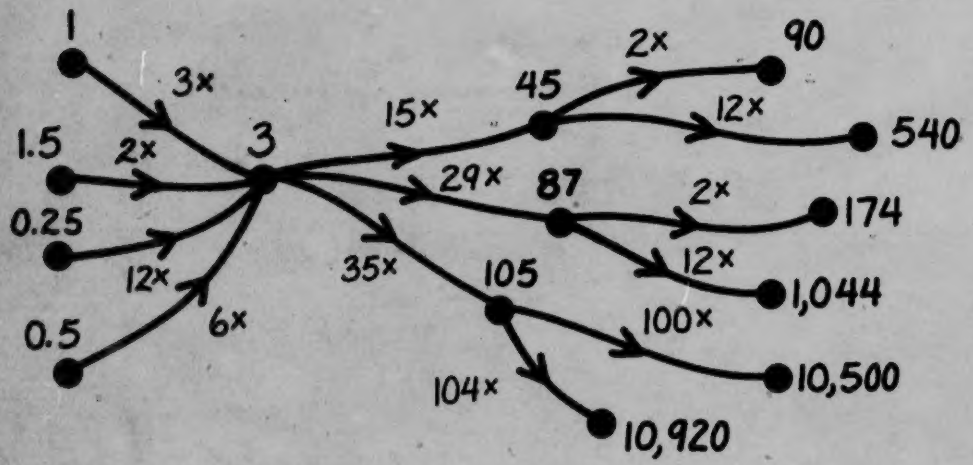
T: Copy the four arrows ending at 3 on your paper. Label the dots as you wish, but be sure each red arrow could be for times a whole number. The dots may be for (non-integer) decimal numbers.

Let students work independently on this problem for a few minutes. Suggest that faster workers add even more arrows ending at 3. Before concluding this activity, invite students to label the dots and the respective arrows at the board.

N 2

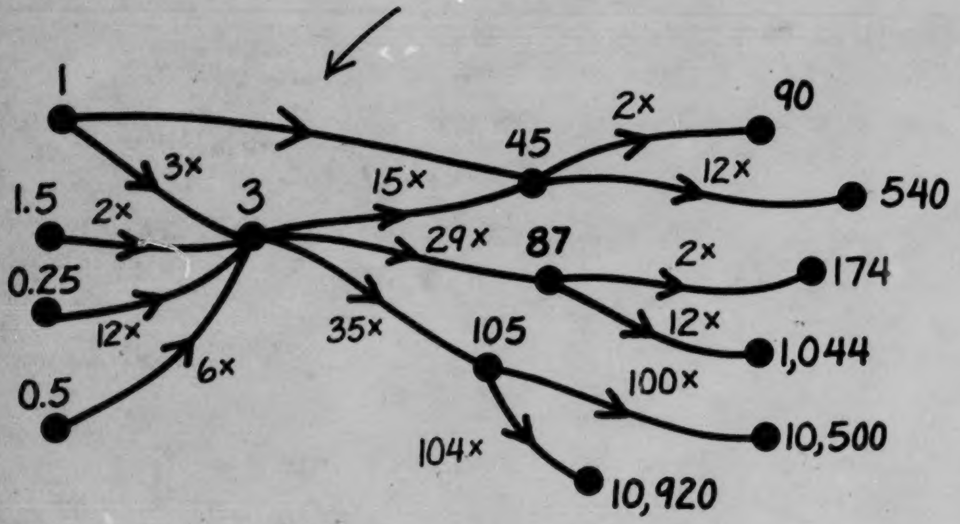
■x

Your completely labeled arrow picture might look like this.


1
3x
1.5
2x
3
0.25
12x
0.5
6x
15x
45
2x
90
12x
540
29x
87
2x
174
12x
1,044
35x
105
100x
10,500
104x
10,920

T: Do you see where we could draw more red arrows between numbers in our picture?

Invite a student to trace an arrow; then you draw it. Suppose an arrow starting at 1 and ending at 45 is drawn.

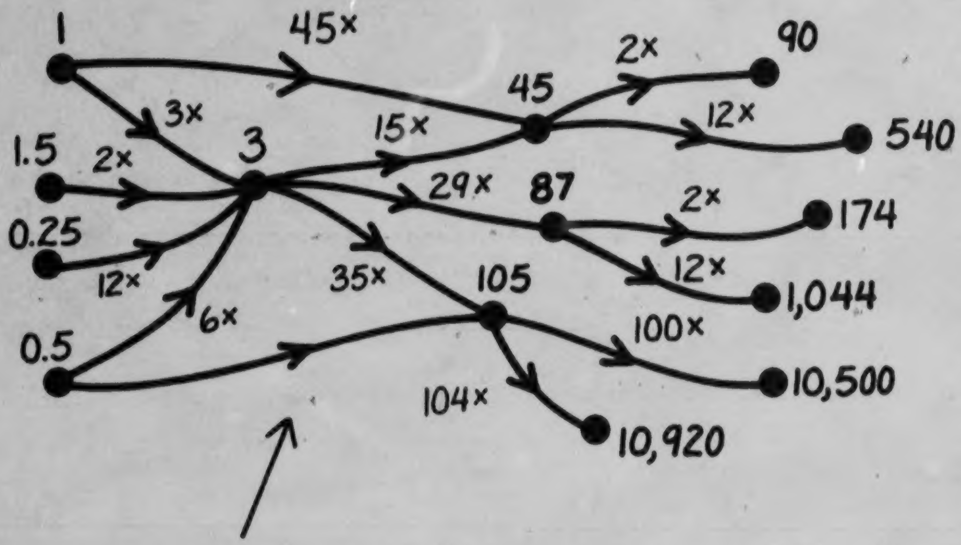

1
3x
1.5
2x
3
0.25
12x
0.5
6x
15x
45
2x
90
12x
540
29x
87
2x
174
12x
1,044
35x
105
100x
10,500
104x
10,920

T: What could this red arrow be for?

S: 45x.

T (tracing the 3x arrow followed by the 15x arrow): Yes. 3x followed by 15x is 45x.

Suppose another arrow starting at 0.5 and ending at 105 is drawn.


1
45x
90
2x
45
3x
12x
540
1.5
2x
3
15x
29x
87
2x
174
0.25
12x
1,044
12x
35x
105
6x
0.5
100x
10,500
104x
10,920

T: What could this red arrow be for?

S: 210x.

T (tracing the 6 x arrow followed by the 35 x arrow): What is 6 × 35?

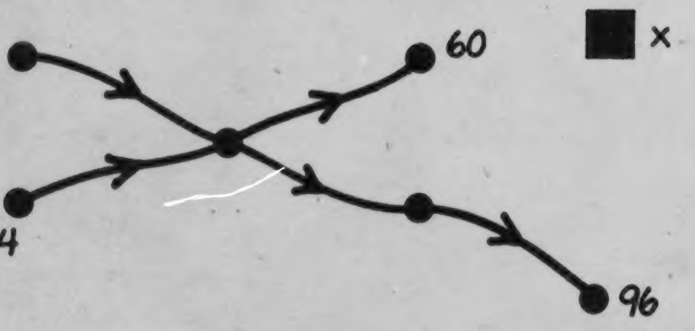
S: 6 × 35 = 210.

T: So 6 x followed by 35 x is 210 x.

N 2

Exercise 2

Draw this arrow picture on the board.

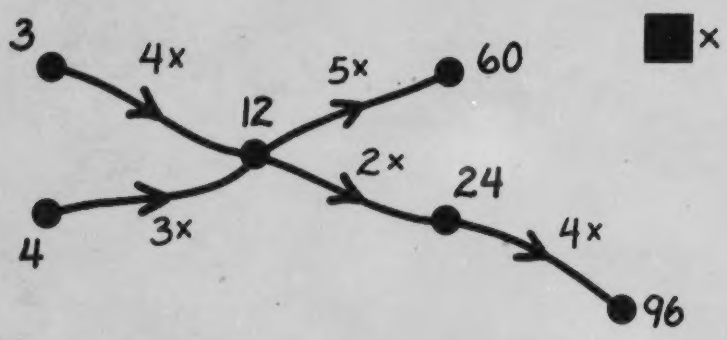


T: Copy this picture on your paper. Label the dots and the arrows. Several answers are possible. Remember, label each arrow to show it could be for times some whole number.

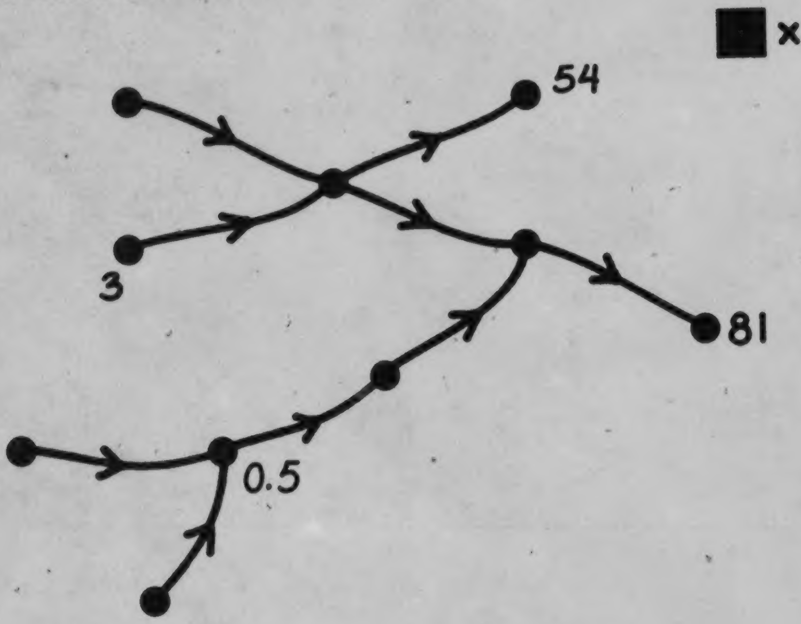
Let the students have several minutes to work independently on this problem. For the faster workers, draw another arrow picture on the board and ask them to label it also. See the example at the end of this lesson.

When most of the students have solved at least one of the problems, ask volunteers to complete the pictures on the board.

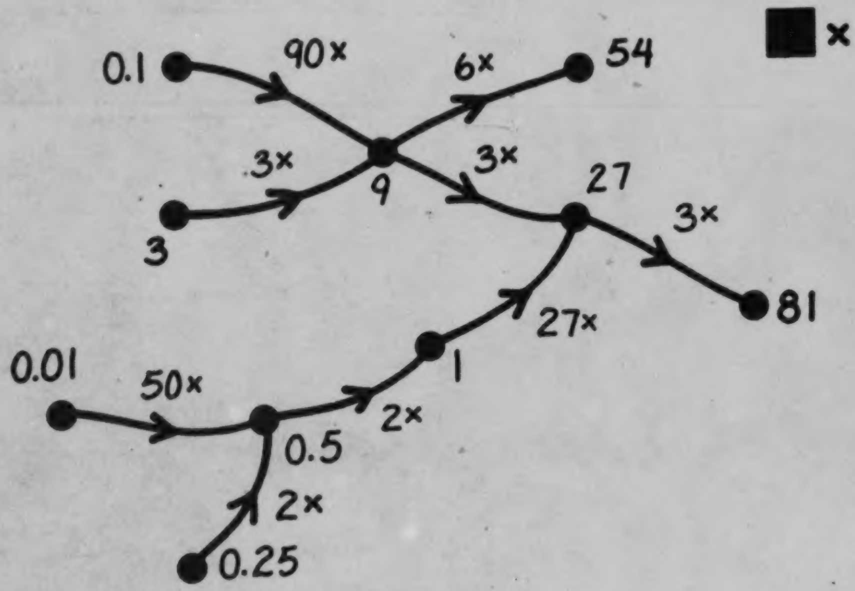
A solution for the first problem is shown below.



Give this second arrow picture to faster workers.



One of the many possible solutions for this problem is shown below.

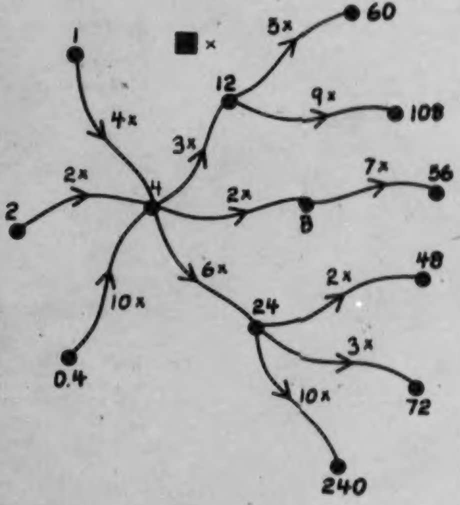

■ x
0.1
90x
6x
54
3x
3x
9
27
3
3x
81
27x
0.01
50x
1
2x
0.5
2x
0.25

Worksheets N05, N06, N07, and N08 are available.

Name ____________ N 05 *

" ■ × " means each red arrow is for times some WHOLE number.

Label the dots. Label each arrow " ☐ × ". Many solutions are possible.

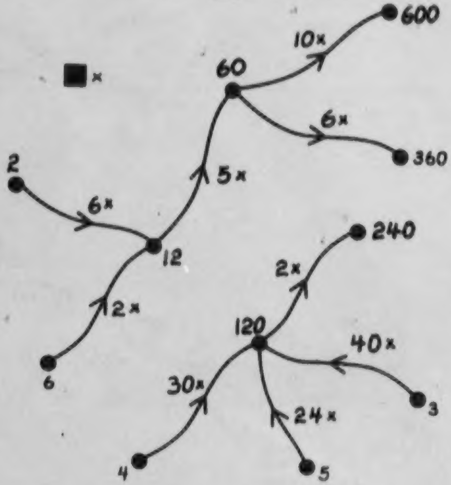

■ ×
1
4×
2
2×
4
10×
0.4
3×
12
5×
60
9×
108
2×
8
7×
56
6×
24
2×
48
3×
72
10×
240

(Many solutions are possible.)

Name ____________ N 06 **

" ■ × " means each red arrow is for times some WHOLE number.

Label the dots. Label each arrow " ☐ × ". Many solutions are possible.

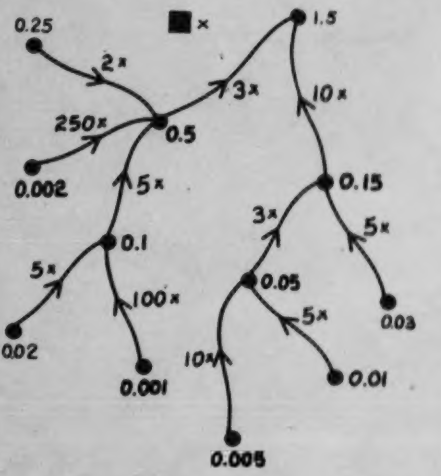

■ ×
600
10×
60
6×
360
5×
2
6×
12
2×
6
240
2×
120
40×
30×
3
24×
4
5

(Many solutions are possible.)

Name ____________ N 07 ***

" ■ × " means each red arrow is for times some WHOLE number.

Label the dots. Label each arrow " ☐ × ". Many solutions are possible.

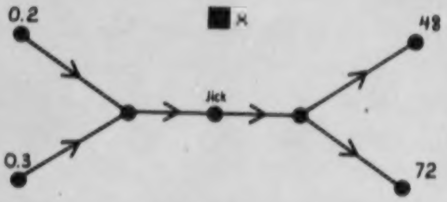

0.25
■ ×
1.5
2×
3×
10×
250×
0.5
0.002
5×
0.15
0.1
3×
5×
5×
100×
0.05
0.03
5×
0.02
0.001
10×
0.01
0.005

(Many solutions are possible.)

Name ____________ N 08 ****

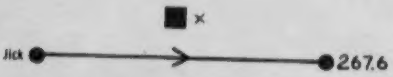
" ■ × " means each red arrow is for times some WHOLE number.

First clue


0.2
■ ×
48
Jick
0.3
72

Jick could be 1.2, 2.4, 3, 4.8, 6, or 12.

Second clue



Jick is 1.2.

1.3 Guess My Rule Games

CAPSULE LESSON SUMMARY

Play a Guess My Rule game with the operation * where $a * b = (3 \times a) + b$. Use this rule for * to do some calculations involving *. Play another Guess My Rule game in which the rule is given by arrows for the function 2x followed by −5. Label the dots in some arrow roads with arrows for this function.

MATERIALS

Teacher: Colored chalk
Student: Paper; pencil; Worksheets N09, N10, N11

DESCRIPTION OF LESSON

Exercise 1

Write this information on the board.

$$3 * 7 = 16$$
$$7 * 3 = 24$$
$$1 * 7 = \boxed{}$$

T: I have a secret rule for * and have given you two clues about my rule. Can you guess my rule? Which number do you think goes in the box?

NOTE: The rule is: $a * b = (3 \times a) + b$. The number in the box is 10 because $1 * 7 = (3 \times 1) + 7 = 10$. Do not tell the class this rule.

Suggest students whisper their guesses in your ear. Acknowledge, aloud, any correct guesses and deny any incorrect ones. For example, "Yes, that's correct." or "No, 14 is not the number." Let a student who guesses correctly tell the class that 10 is in the box. If no one guesses correctly, put a 10 in the box yourself, but do not give away the rule.

N 11

Continue the activity with these problems (answers are in the boxes).

$$7 * 1 = \boxed{22}$$

$$4 * 0 = \boxed{12}$$

$$0 * 4 = \boxed{4}$$

$$10 * 6 = \boxed{36}$$

T: Can you explain my rule?

S: The secret number is 3 times the first number plus the second number. For example, $10 * 6 = 36$ because $(3 \times 10) + 6 = 36$.

Write the rule on the board.

$$\boxed{a * b = (3 \times a) + b}$$

T: Copy the problems I write on the board and use the rule to solve these problems.

Write these problems on the board. Answers are in the boxes; do not write them on the board.

$$6 * 10 = \boxed{28}$$

$$2 * 0.5 = \boxed{6.5}$$

$$0.5 * 2 = \boxed{3.5}$$

$$(2 * 3) * 4 = \boxed{31}$$

$$2 * (3 * 4) = \boxed{19}$$

$$6 * \boxed{12} = 30$$

$$\boxed{8} * 6 = 30$$

$$10 * \boxed{^-5} = 25$$

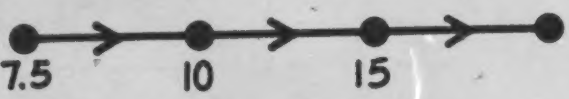
$$\boxed{5} * 10 = 25$$

Check the solutions to these problems collectively.

Exercise 2

T: I have a new rule for you to guess, but this time my rule is for red arrows.

Slowly draw this arrow road on the board.

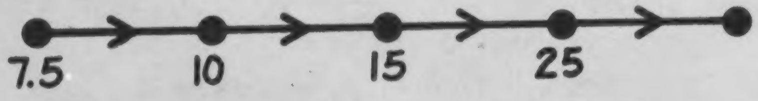


NOTE: The red arrows are for the function 2x followed by -5. For example, the red arrow starting at 15 ends at 25 because (2 x 15) - 5 = 25. Do not tell the class this rule.

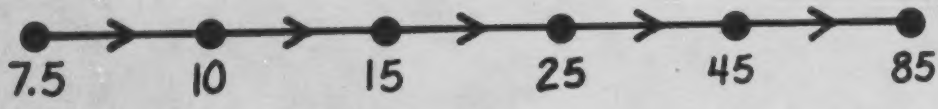
Give the students an opportunity to guess what the ending number could be. If someone guesses correctly, say:

T: Very good. Maybe you know what my rule is. Don't tell if you do; see if you can use it to label another dot.

Label the appropriate dot "25" (if necessary, tell the class this ending number) and add another arrow starting at 25.



Again give the students a chance to tell you their guesses for the ending number. Ask them to write it on their papers or whisper it to you. Let a student tell the class how to label the dot (if necessary, tell them yourself) and then label the dot "45". Continue by adding another arrow to the road and asking the students for the ending number.



40

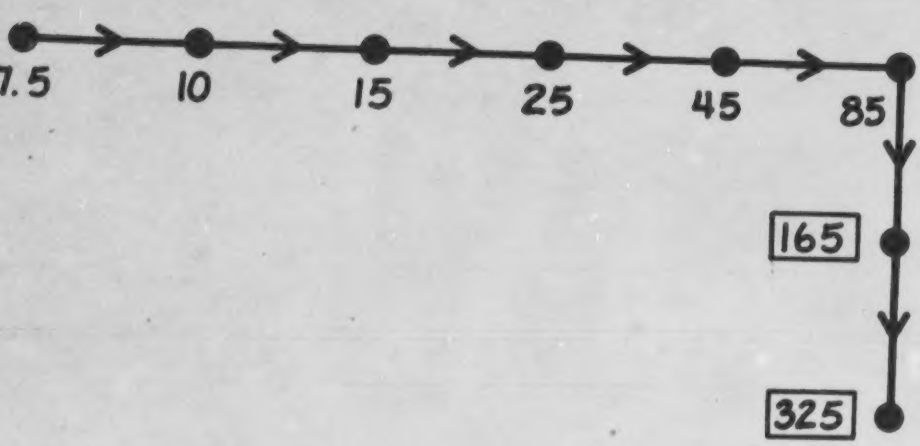
N 11

T: Who can explain my rule?

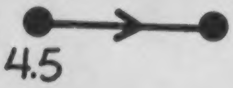
S: The ending number of an arrow is 2 times the starting number minus 5.

Write the rule on the board as shown below. Add two more arrows to the arrow road and ask the students to label the dots. Answers are in the boxes.

$$\boxed{c \longrightarrow (2 \times c) - 5}$$



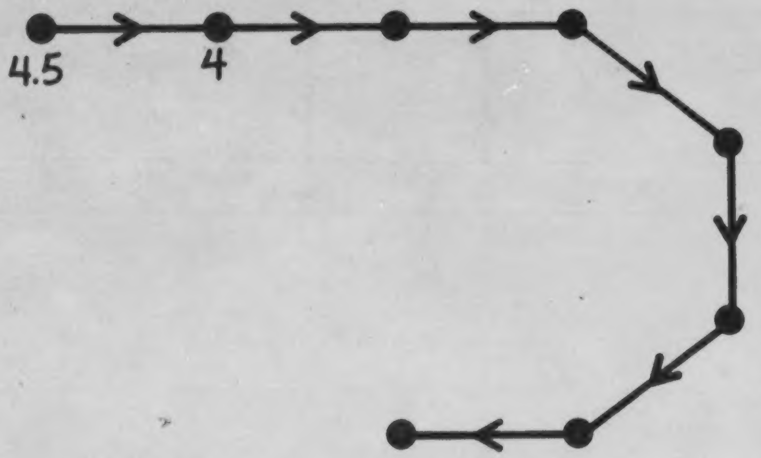
Start a new road by drawing this arrow picture on the board.



T: What must the ending number be? Why?

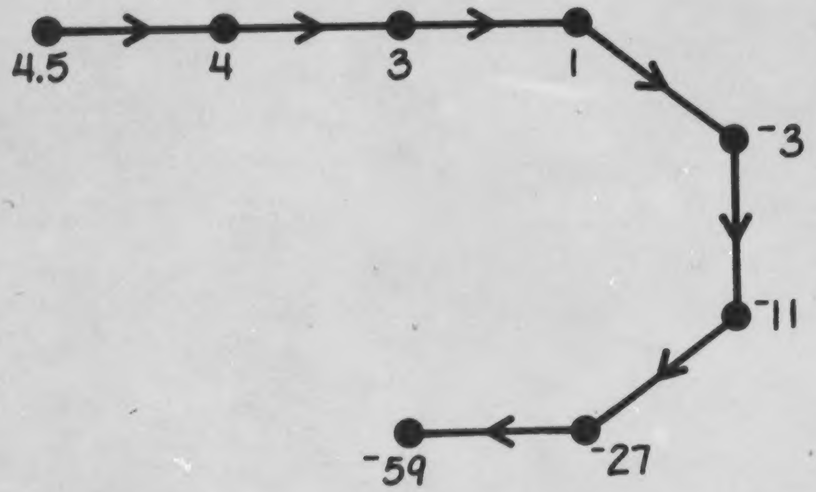
S: 4, because (2 x 4.5) - 5 = 9 - 5 = 4.

Label the dot "4" and extend the road.



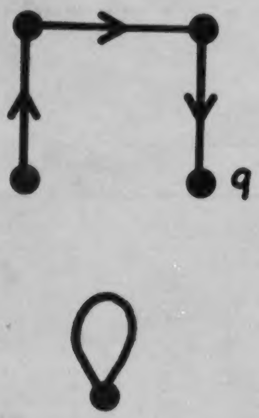
T: Copy this picture on your paper and label the dots.

Let the students have several minutes to work independently while you observe individual student progress. Collectively, complete the picture on the board as shown below.

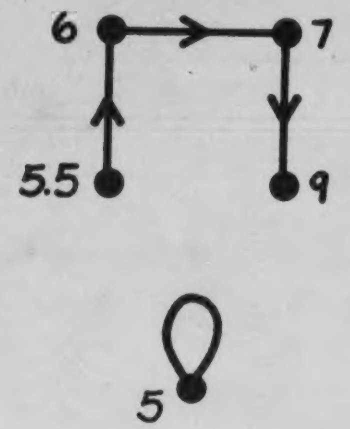


N 11

Draw this picture on the board and ask the students to copy it and label the dots.



After the students have had several minutes to work, collectively complete the picture as shown below.



Worksheets N09, N10, and N11 are available.

Name ____________ N 09 *

$a * b = (3 \times a) - b$

Complete:

$5 * 4 = 11$

$4 * 5 = \underline{7}$

$10 * 4 = \underline{26}$

$4 * 10 = \underline{2}$

$7 * 6 = \underline{15}$

$6 * 7 = \underline{11}$

$8 * 9 = \underline{15}$

$9 * 8 = \underline{19}$

$15 * 5 = \underline{40}$

$5 * 15 = \underline{0}$

$5 * \boxed{3} = 12$

$6 * \boxed{4} = 14$

$7 * \boxed{5} = 16$

$8 * \boxed{6} = 18$

$9 * \boxed{7} = 20$

$\boxed{4} * 5 = 7$

$\boxed{6} * 8 = 10$

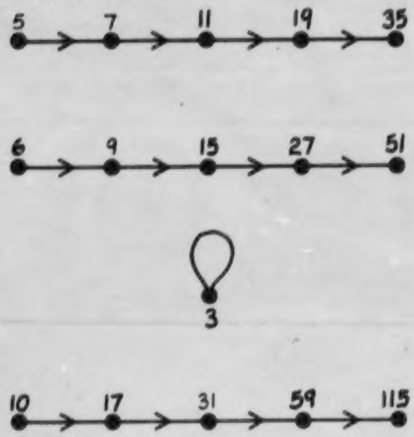
$\boxed{8} * 10 = 14$

$\boxed{10} * 12 = 18$

Name ____________ N 10 **

$a \rightarrow (2 \times a) - 3$

Label the dots.



Name ____________ N 11 ***

$a * b = a + (b \div 3)$

Complete:

$15 * 12 = 19$

$12 * 15 = \underline{17}$

$60 * 27 = \underline{69}$

$27 * 60 = \underline{47}$

$0.6 * 1.5 = \underline{1.1}$

$3.4 * 3.6 = \underline{4.6}$

$\boxed{7} * 24 = 15$

$\boxed{6} * 42 = 20$

$\boxed{6} * 72 = 30$

$2 * \boxed{15} = 7$

$0 * \boxed{21} = 7$

$1 * \boxed{27} = 10$

$5 * \boxed{45} = 20$

$8 * \boxed{0} = 8$

$3 * \boxed{^{-}9} = 0$

$^{-}3 * \boxed{9} = 0$

$^{-}12 * \boxed{30} = {}^{-}2$

N 12

1.4 Multiplication #2

CAPSULE LESSON SUMMARY

Progressively build an arrow picture with various multiplication and division functions. Locate places where arrows for other multiplication functions (composites of arrows already in the picture) could be drawn. Perform some multiplication calculations using addition and subtraction and results of other related calculations. Observe some patterns in a certain sequence of calculations.

MATERIALS

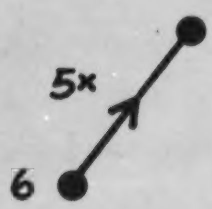
Teacher: Colored chalk
Student: Paper; pencils; Worksheets N12, N13, N14

DESCRIPTION OF LESSON

Exercise 1

T: I shall draw an arrow picture on the board. Each time I draw an arrow you must calculate the ending number.

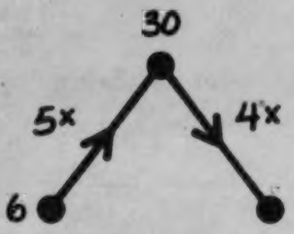
Begin the arrow picture with this arrow.



T (pointing to the ending dot): Which number is here?

S: 30.

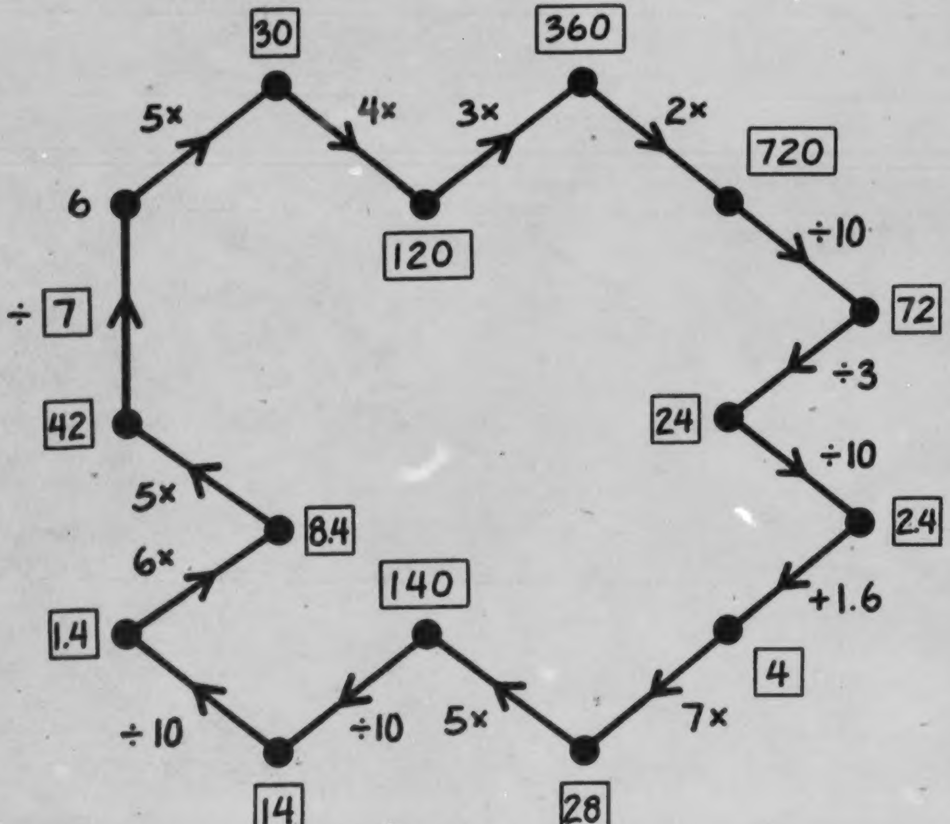
Label the dot "30" and extend the picture.



T (pointing to the ending dot): Which number is here?

S: 120.

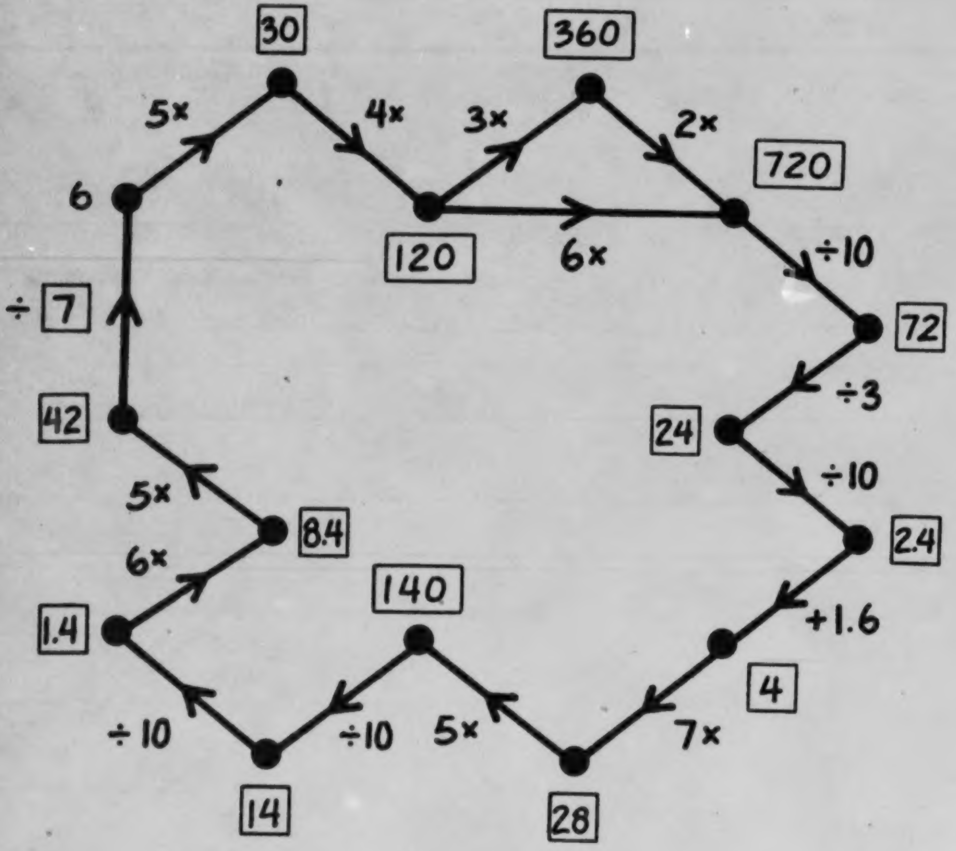
Label the dot "120". Continue extending the arrow picture and asking students to calculate ending numbers. The arrow picture below with correct responses boxed is for your use. Feel free to adjust the calculations to the level of your class or to stop the activity if students lose interest.


30
360
5×
4×
3×
2×
720
6
120
÷10
÷7
72
÷3
42
24
÷10
5×
8.4
2.4
6×
140
+1.6
1.4
4
÷10
÷10
5×
7×
14
28

N 12

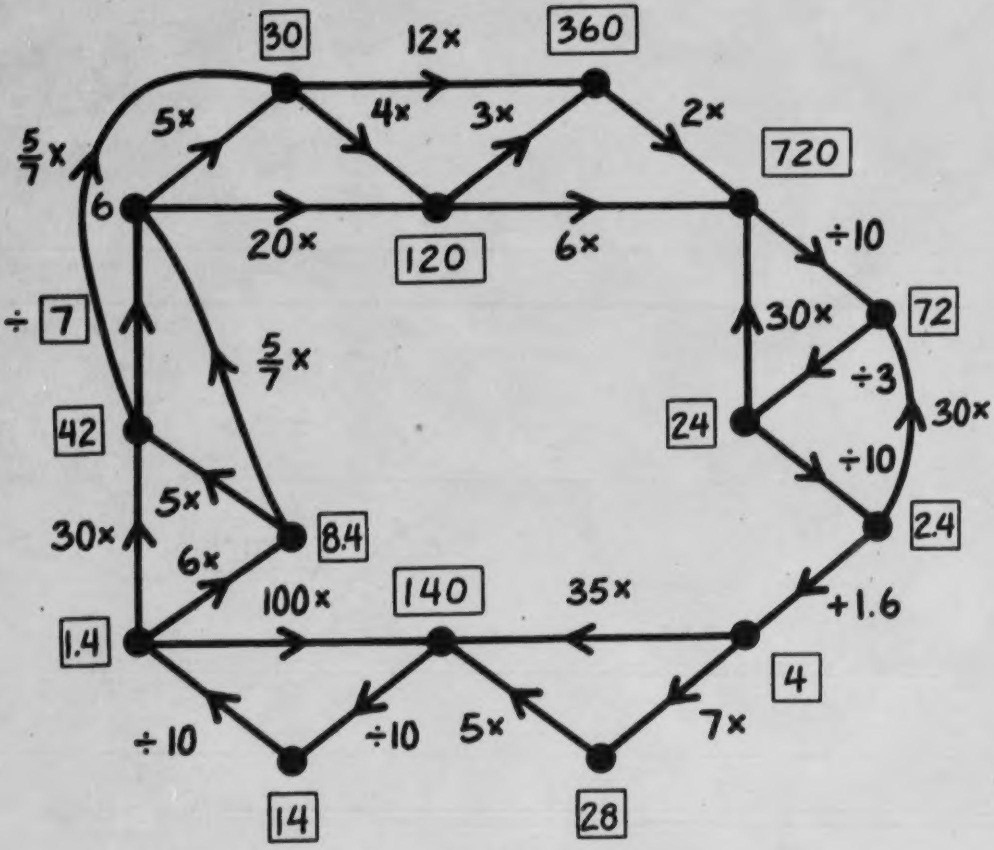
T: Who can trace a (new) 6x arrow?

Invite a student to trace a 6x arrow; it is not necessary to draw the arrow, but do encourage the student to note that 3x followed by 2x is 6x.



30
360
5x
4x
3x
2x
720
6
120
6x
÷10
÷ 7
72
÷3
42
24
÷10
5x
8.4
2.4
6x
140
+1.6
1.4
4
÷ 10
÷10
5x
7x
14
28

Ask the students to trace arrows for 20x; 30x; 35x; 100x; and $\frac{5}{7}$x. These arrows are included in the following illustration.


30
12×
360
5/7 ×
5×
4×
3×
2×
720
6
20×
120
6×
÷10
÷ 7
30×
72
5/7 ×
÷3
42
24
30×
÷10
5×
24
30×
8.4
6×
2.4
100×
140
35×
+1.6
1.4
4
÷ 10
÷10
5×
7×
14
28

Exercise 2

Put the calculation boxed in red on the board and call the class' attention to it. Then write two related problems under the given calculation.

$$\boxed{13 \times 25 = 325}$$

$12 \times 25 = \square$

$14 \times 25 = \square$

N 12

T: Can you solve these problems mentally using the fact that $13 \times 25 = 325$?

Let the students have a minute to consider the problems.

S: $12 \times 25 = 300$; just subtract 25 from 13×15.

S: $14 \times 25 = 350$; just add 25 to 13×25.

Summarize the student's explanation by writing this information on the board.

$$12 \times 25 = 325 - 25 = \boxed{300}$$

$$14 \times 25 = 325 + 25 = \boxed{350}$$

Write two more problems on the board and ask the students to solve them mentally. Encourage them to use only addition and subtraction and results of previous problems.

$$14 \times 26 = \boxed{} \qquad 13 \times 24 = \boxed{}$$

Ask students to explain their methods of doing the calculations mentally and summarize a method by writing a calculation on the board. For example,

$$14 \times 26 = 350 + 14 = 364$$

$$13 \times 24 = 325 - 13 = 312$$

Continue in the same way with three more problems.

$$23 \times 25 = \boxed{}$$

$$13 \times 35 = \boxed{}$$

$$113 \times 25 = \boxed{}$$

Invite the students to explain their methods for solving these problems.

$$23 \times 25 = 325 + 250 = \boxed{575}$$

$$13 \times 35 = 325 + 130 = \boxed{455}$$

$$113 \times 25 = 325 + 2{,}500 = \boxed{2{,}825}$$

Exercise 3

Write these problems on the board and ask the students to do the calculations on their papers. Answers are in the boxes.

$$13 \times 7 = \boxed{91}$$

$$14 \times 8 = \boxed{112}$$

$$15 \times 9 = \boxed{135}$$

$$16 \times 10 = \boxed{160}$$

$$17 \times 11 = \boxed{187}$$

$$18 \times 12 = \boxed{216}$$

$$19 \times 13 = \boxed{247}$$

$$20 \times 14 = \boxed{280}$$

After a couple of minutes, invite students to fill in the boxes on the board.

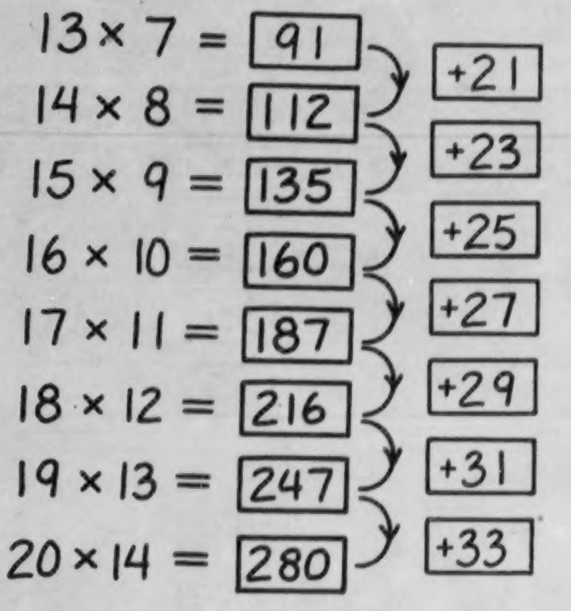
T: Do you notice anything interesting about these problems?

S: There is a pattern; start with "13 x 7" and add 1 to each factor to get "14 x 8" and so on.

T: Yes, do you see any pattern in the answers?

S: The second answer is 21 more than the first, and the third answer is 23 more than the second, and the fourth answer is 25 more than the third, and so on.

Add these arrows to your list and let the students label them. Answers are in the boxes



13 × 7 = 91
14 × 8 = 112
15 × 9 = 135
16 × 10 = 160
17 × 11 = 187
18 × 12 = 216
19 × 13 = 247
20 × 14 = 280
+21
+23
+25
+27
+29
+31
+33

T: How can we explain this pattern?

Let the students make any comments they wish. More than likely, you will need to direct the discussion of this explanation.

T: Let's consider an example, say 19 × 13. Let's see if we can find a reason for 19 × 13 to be 31 larger than 18 × 12. The key is 19 × 12.

- Write this information on the board with blank boxes and ask the students to fill in the boxes.

$$\begin{tikzcd}[row sep=small] 18 \times 12 = 216 \arrow[d, bend left] & \\ 19 \times 12 = \boxed{228} \arrow[d, bend left] & +\boxed{12} \\ 19 \times 13 = 247 & +\boxed{19} \end{tikzcd}$$

T: How much larger is 19 x 13 than 18 x 12? Why?

S: 31 larger because +12 followed by +19 is +31.

S: You can get 31 another way; 12 + 18 + 1 = 31.

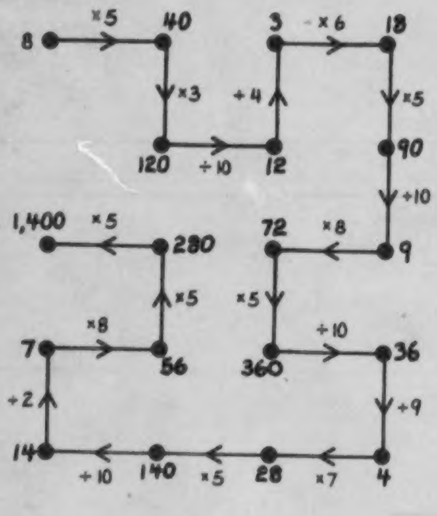
S: It works the same to go from 13 x 7 to 14 x 8; just add 7 + 13 + 1 = 21.

Accept any comments the students might make.

Worksheets N12, N13, and N14 are available.

Name ____________________ N 12 *

Label the dots.



Name ____________________ N 13 **

Complete each problem. Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each box to make the calculations correct. You may use a digit more than once.

$$\begin{array}{r} 68 \\ \times 42 \\ \hline 136 \\ 2{,}720 \\ \hline 2{,}856 \end{array}$$

$$\begin{array}{r} 98 \\ \times 2\boxed{4} \\ \hline 3\boxed{9}\boxed{2} \\ \boxed{1}\boxed{9}\boxed{6}\boxed{0} \\ \hline \boxed{2}\boxed{3}\boxed{5}\boxed{2} \end{array}$$

$$\begin{array}{r} \boxed{3}\boxed{2}7 \\ \times \boxed{2}6 \\ \hline 196\boxed{2} \\ \boxed{6}\boxed{5}\boxed{4}0 \\ \hline \boxed{8}\boxed{5}0\boxed{2} \end{array}$$

$$\begin{array}{r} 137 \\ \times 59 \\ \hline 1{,}233 \\ 6{,}850 \\ \hline 8{,}083 \end{array}$$

Name ____________________ N 14 ***

$16 \times 24 = 384$

Complete:

$17 \times 24 = 384 + \boxed{24} = \underline{408}$

$15 \times 24 = 384 - \boxed{24} = \underline{360}$

$16 \times 25 = 384 + \boxed{16} = \underline{400}$

$16 \times 23 = 384 - \boxed{16} = \underline{368}$

$26 \times 24 = 384 + \boxed{240} = \underline{624}$

$16 \times 34 = 384 + \boxed{160} = \underline{544}$

$116 \times 24 = 384 + \boxed{2{,}400} = \underline{2{,}784}$

$16 \times 124 = 384 + \boxed{1{,}600} = \underline{1{,}984}$

1.5 Sums of Sequences of Whole Numbers

CAPSULE LESSON SUMMARY

Develop short cut methods for adding certain sequences of whole numbers. Observe that the sum of the first n odd whole numbers is a square number, n x n. (Optional) Find an explanation for this result.

MATERIALS

Teacher: Colored chalk
Student : Paper; pencil

DESCRIPTION OF LESSON

NOTE: To allow for a wide range in students' abilities, two options have been built into this lesson. With an average to above average class, begin with Exercise 2 and proceed to Exercises 3 and 4. With a below average class, begin with Exercise 1 and proceed to Exercise 2 and then to Exercise 4 (omitting Exercise 3).

Exercise 1

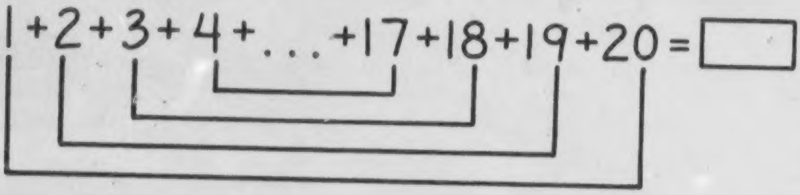
Write this problem on the board.

$$1+2+3+4+\ldots+17+18+19+20=\boxed{}$$

T: Let's try to find the sum of the numbers from 1 to 20. I have written the problem on the board. The three dots indicate that the numbers between 4 and 17 are also to be added into the sum although I have omitted writing that part of the problem.

Can you think of an easy way to do this addition problem?

Give the class a minute or two to consider the problem. If no suggestions for short-cuts are forthcoming, begin to pair numbers as shown in the illustration. Elicit student comments about what you are doing.



$$1+2+3+4+\ldots+17+18+19+20=\boxed{}$$

S: You can continue to pair numbers until you get to "10 + 11".

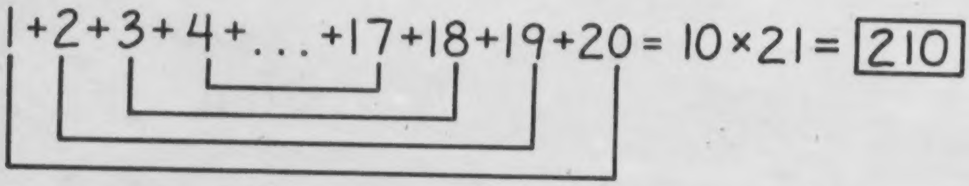
S: Each pair adds to 21.

T: How many pairs of numbers like this are there?

S: There are twenty numbers, so there are ten pairs of numbers.

S: 10 x 21 = 210. The sum is 210.

Complete the calculation on the board.



$$1+2+3+4+\ldots+17+18+19+20=10\times21=\boxed{210}$$

T: If we pair the numbers, then we can use multiplication to find the sum.

Write a second problem on the board beneath the first problem.

$$1+3+5+\ldots+15+17+19=\boxed{}$$

T: Can you find an easy way to add the odd numbers from 1 to 19?

For the benefit of those students who need a hint, ask questions like the following:

T: Do you see a way to pair numbers like we did in the first problem?

S: Pair 1 and 19, 3 and 17, 5 and 15, and so on. Each time the sum of a pair of numbers is 20.

Indicate the pairing on the board as a student describes it.

T: How many odd numbers from 1 to 20 are there?

S: Ten, every other number is odd.

T: Ten odd numbers. How many pairs of odd numbers are there?

S: Five pairs.

T: What is the sum of each pair?

S: 20.

T: Use this information to do the problem quickly.

S: 5 x 20 = 100, so the solution is 100.

Complete the calculation on the board.

$$1+2+3+4+\ldots+17+18+19+20=10\times 21=\boxed{210}$$

$$1+3+5+\ldots+15+17+19=5\times 20=\boxed{100}$$

N 23

T: Now, find the sum of the even numbers.

Write a third problem on the board beneath the other two.

$$2+4+6+\ldots+16+18+20=\boxed{}$$

T: Can you find an easy way to add the even numbers from 2 to 20?

One way the class might suggest is to use the pairing method just introduced. In that case, proceed as above. However, another method might be suggested.

S: I subtracted. The sum of the first twenty whole numbers is 210 and the sum of the first ten odd whole numbers is 100, so the sum of the first ten even whole numbers is 210 - 100 or 110.

Direct the discussion so that the class observes both methods.

Complete the calculation on the board.

$$1+2+3+4+\ldots+17+18+19+20=10\times21=\boxed{210}$$

$$1+3+5+\ldots+15+17+19=5\times20=\boxed{100}$$

$$2+4+6+\ldots+16+18+20=5\times22=210-100=\boxed{110}$$

T: Which of these sums is a square number?

S: 100, because 10 x 10 = 100.

S: 110 is not a square number, because 10 x 10 = 100 and 11 x 11 = 121 and there are no whole numbers between 10 and 11.

T: What about 210? Can you convince me one way or the other?

This is a more challenging question. Quite likely the argument that 210 is not a square number will require some direction from you.

Write this information on the board.

$$\square \times \square = 210$$

T: Suppose 210 is a square number. Can you estimate which whole number $\square$ could be?

S: 10 x 10 = 100 and 20 x 20 = 400, so $\square$ is a number between 10 and 20.

T: Which number should we try?

S: 15.

Perhaps many students will immediately want to calculate "15 x 15" on paper. Instead, encourage them to focus on the ones' digit of the product.

T: When we calculate "15 x 15", which number will be the ones' digit?

S: 5 will be the last digit, so 15 x 15 cannot equal 210.

S: If we square any number between 10 and 20, we will not get a number ending in 0.

T: Very good; that means that 210 is not a square number.

Exercise 2

NOTE: Even if you omit Exercise 1 in class, read it to acquaint yourself with the presentation. This exercise is presented in a similar way.

Write this problem on the board.

$$1+2+3+\ldots+98+99+100=\boxed{}$$

T: Find the sum of these 100 whole numbers.

Proceed with this exercise in a manner similar to that of Exercise 1. The students should use the pairing technique to conclude:

$$1+2+3+\ldots+98+99+100 = 50\times 101 = \boxed{5{,}050}$$

Direct the class to find the sum of the first 50 even whole numbers and the first 50 odd whole numbers. The results are summarized below.

$$1+3+5+\ldots+95+97+99 = 25\times 100 = \boxed{2{,}500}$$

$$2+4+6+\ldots+96+98+100 = 25\times 102$$

$$= 5{,}050 - 2{,}500 = \boxed{2{,}550}$$

T: Look carefully at the three results. Which of the three numbers is a square number?

S: 2,500 is a square number (50 x 50 = 2,500).

T: What about 5,050 and 2,550?

S: 2,550 is between 50 × 50 and 60 × 60, because 50 × 50 = 2,500 and 60 × 60 = 3,600.

S: If we square any number between 50 and 60, we will not get a number ending in 0.

T: Very good; 2,550 is not a square number. What about 5,050?

S: 5,050 is between 70 × 70 and 80 × 80, because 70 × 70 = 4,900 and 80 × 80 = 6,400.

S: 5,050 is not a square number. If we square any number between 70 and 80, we will not get a number ending in 0.

Exercise 3

Ask the students to solve these problems on their paper.

$$1+2+3+4+\dots+397+398+399+400=\boxed{}$$

$$1+3+5+\dots+395+397+399=\boxed{}$$

$$2+4+6+\dots+396+398+400=\boxed{}$$

When most of the class have finished, invite some students to explain their calculations. Answers are provided below.

$$1+2+3+4+\dots+397+398+399+400=200\times401=\boxed{80{,}200}$$

$$1+3+5+\dots+395+397+399=100\times400=\boxed{40{,}000}$$

$$\begin{aligned}2+4+6+\dots+396+398+400&=100\times402\\&=80{,}200-40{,}000=\boxed{40{,}200}\end{aligned}$$

T: Which of these sums is a square number?

S: 40,000; because 200 x 200 = 40,000.

Exercise 4

T: Each time, the sum of the odd numbers has been a square number. Do you think this will always be true?

Let the class discuss this situation. Perhaps someone will suggest looking at some other sums of odd numbers.

T: Calculate "1 + 3 + 5 + 7".

S: 16 or 4 x 4.

S: When we add the first four odd (whole) numbers, we get the square number 4 x 4.

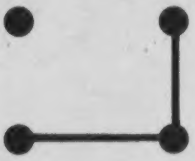
If this last observation is not made by a student, make it yourself.

T: Perhaps a picture will help explain this situation.

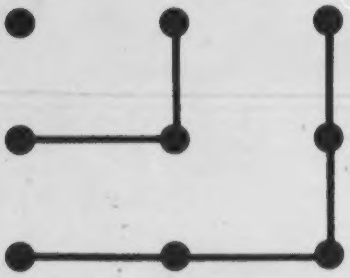
N 23

Progressively draw this picture on the board.

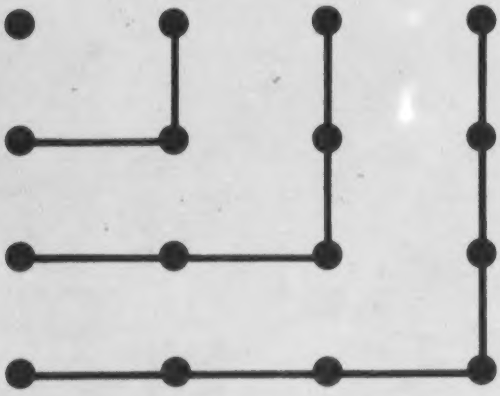
$1 = 1 \times 1$



$1 + 3 = 4 = 2 \times 2$



$1 + 3 + 5 = 9 = 3 \times 3$



$1 + 3 + 5 + 7 = 16 = 4 \times 4$

S: Each time adding a "layer" of dots is like adding the next odd number.

S: Each time an odd number is added, the square gets bigger.

Accept any reasonable comments the students make.

T: Now let's reverse the situation. Give me a square number and let's try to write it as a sum of odd numbers.

S: 64 is a square number ($8 \times 8 = 64$).

Write this information on the board.

$$64 = 1 + 3 + 5 + \ldots + \boxed{}$$

T: How can we fill in the box?

S: $64 = 8 \times 8$, so we need eight odd numbers.

S: 15 is in the box, $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 64$.

Complete the problem on the board.

$$64 = 1 + 3 + 5 + \ldots + \boxed{15} = 8 \times 8$$

Ask the students to write several more square numbers as sums of odd numbers. Write the problems that you use on the board; for example,

$$25 = 1 + 3 + 5 + \ldots + \boxed{}$$
$$144 = 1 + 3 + 5 + \ldots + \boxed{}$$
$$225 = 1 + 3 + 5 + \ldots + \boxed{}$$

Collectively, check the results of these problems.

$$64 = 1 + 3 + 5 + \ldots + \boxed{15} = 8 \times 8$$

$$25 = 1 + 3 + 5 + \ldots + \boxed{9} = 5 \times 5$$

$$144 = 1 + 3 + 5 + \ldots + \boxed{23} = 12 \times 12$$

$$225 = 1 + 3 + 5 + \ldots + \boxed{29} = 15 \times 15$$

T: Do you notice anything interesting about the numbers in the boxes and the corresponding multiplication problem?

Give the class a minute to consider this situation.

S: Look at 64. (2 x 8) - 1 = 15 and 15 is the number in the box.

S: That works for each of the examples.

Accept any reasonable explanation even if it is not exactly like this one.

2

MULTIPLES AND DIVISORS

This chapter provides extended experiences with multiples and divisors of numbers in situations involving hand-calculator relations, such as $\boxed{+}\ \boxed{7}\ \boxed{=}$. . . .* It introduces the notions of common multiples and least common multiple and of common divisors and greatest common divisor. First these notions appear intuitively in exploratory situations involving combinations of hand-calculator relations and intersections of strings. Problems involving adding, for example, two multiples of 3 or two non-multiples of 3, and patterns generated by moving one checker in a configuration on the Minicomputer suggest interesting properties of multiples. Guess My Rule games introduce explicitly the least common multiple and greatest common divisor relations, and introduce a new relation, $=_3$. Detective stories, arrow diagrams and string pictures provide opportunities to explore the new relations and to discover some properties of them.

CAPSULE LESSON SUMMARIES

Section 1.1: Solve a detective story in which the clues involve a composition of the hand-calculator relations $\boxed{+}\ \boxed{3}\ \boxed{=}$. . . and $\boxed{-}\ \boxed{10}\ \boxed{=}$. . . , positive divisors of 18, and a pattern generated by using the hand-calculator relation $\boxed{-}\ \boxed{5}\ \boxed{=}$. . . .
(Worksheets N15, N16)

Section 2.2: Solve a detective story in which clues involve patterns generated by combined use of the hand-calculator relations $\boxed{+}\ \boxed{3}\ \boxed{=}$. . . and $\boxed{+}\ \boxed{7}\ \boxed{=}$. . . and combined use of $\boxed{+}\ \boxed{2}\ \boxed{=}$. . . and $\boxed{+}\ \boxed{5}\ \boxed{=}$. . . , and in which the final clue is a three-string picture.
(Worksheets N17, N18)

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* See Section 2 (f) of the General Introduction for a discussion of the constant mode for hand-calculatiors.

Section 2.3: Play Guess My Rule games using greatest common divisor and least common multiple as the secret operations. That is, the rule for one game is: $a * b$ is the greatest common divisor of a and b. For another game, the rule is: $a * b$ is the least common multiple of a and b.

Section 2.4: Examine characteristics of the sum of two multiples of 3, the sum of a multiple of 3 and a non-multiple of 3, and the sum of two non-multiples of 3. Solve a detective story in which the clues involve multiples of 3, patterns generated by moving one checker in a configuration on the Minicomputer, numbers which can be put on the ones' board of the Minicomputer with weighted checkers, and a string picture. (Worksheets N19, N20, N21, N22)

Section 2.5: Review what is meant by the greatest common divisor (gcd) and the least common multiple (lcm) of two numbers. Introduce the notation "$\sqcap$" for gcd and "$\sqcup$" for lcm. Explore relationships between statements about $\sqcap$ ($\sqcup$) and string pictures with strings for positive divisors (multiples). Present a detective story where the clues involve $\sqcap$, prime divisors, and a string picture where the strings are for positive divisors of certain numbers.

Section 2.6: Play a Guess My Rule game with a secret rule for arrows. The secret rule is: there is an arrow from x to 0 if x is a multiple of 3, from x to 1 if x is 1 more than a multiple of 3, and from x to 2 if x is 2 more than a multiple of 3. With the aid of a string picture, explore the possibilities when you add two numbers which are not multiples of 3. Use the string picture to introduce $=_3$ (read "equals modulo 3") and observe that two numbers are equal modulo 3 if and only if their difference is a multiple of 3. (Worksheets N23, N24, N25)

2.1 Who is Paf?

CAPSULE LESSON SUMMARY

Solve a detective story in which the clues involve a composition of the hand-calculator relations $\boxed{+}\boxed{3}\boxed{=}$ ... and $\boxed{-}\boxed{10}\boxed{=}$ ..., positive divisors of 18, and a pattern generated by using the hand-calculator relation $\boxed{-}\boxed{5}\boxed{=}$ ... .

MATERIALS

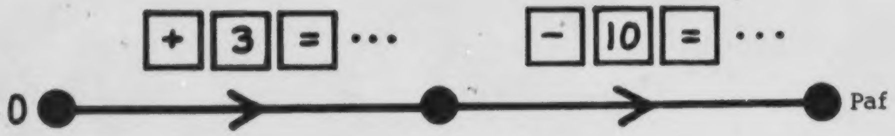
Teacher: Colored chalk
Student: Hand-calculator; pencil; Worksheets N15, N16

DESCRIPTION OF LESSON

NOTE: The first clue of this detective story may be investigated on two different levels depending on the numerical ability of your class. Weaker students might consider only the pattern that the arrow picture puts in evidence, while stronger students might try to find an explanation for the pattern.

First clue

Draw this arrow picture on the board and distribute hand-calculators.



T: Which numbers could Paf be?

Let the students have a few minutes to explore this situation.

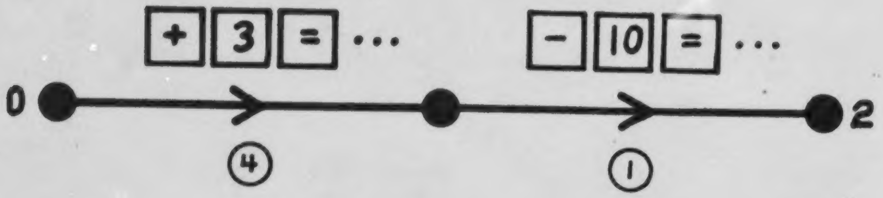
S: Paf could be 2.

T: Explain why Paf could be 2.

N 3

S: Press $\boxed{+}$ $\boxed{3}$ and $\boxed{=}$ four times to get 12, then press $\boxed{-}$ $\boxed{10}$ and $\boxed{=}$ once to get 2.

As a student is giving an explanation of this kind, write the information in the arrow picture in this way.



Continue by asking for other numbers that Paf could be. It is not necessary to elicit an explanation for every suggested number that Paf could be; however, do list them on the board. After a few minutes you should have a list of eight or ten possibilities for Paf. Since Paf can be any integer except 0, we cannot predict which numbers your students will suggest, but suppose you have this list.

$^{-}867$, $^{-}7$, $^{-}2$, 2, 5, 8, 38, 44

Encourage the students to fill in the "gaps". For example,

T: Could Paf be 1?

S: Yes. Press $\boxed{+}$ $\boxed{3}$ and $\boxed{=}$ seven times followed by $\boxed{-}$ $\boxed{10}$ and $\boxed{=}$ twice.

Include 1 in your list.

T: What about 3 and 4?

S: Paf could be 3; $11 \times 3 = 33$ and $33 - 30 = 3$.

S: Paf could be 4; 8 × 3 = 24 and 24 - 20 = 4.

T: Try to find 6, 7, 9, and 10 by yourself.

Continue to find other numbers that Paf could be as long as there is interest. At some point during this discussion ask if there are integers which Paf cannot be. If no comments are forthcoming, do not worry as there will be other opportunities to explore this situation.

When the class is ready, continue with the second clue or lead a discussion which explains why Paf could be any integer except 0.

T: We have found that Paf could be many of the positive integers. Do you think that Paf could be any positive integer?

Let the students react to the question and listen to their arguments.

T (pointing to the middle dot): One way of understanding is to find a pattern for numbers that could be here.

S: 3 could be there.

S: Or 6.

S: 12.

S: Any positive multiple of 3.

Continue listing these multiples of 3 until you have this list.

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36

T: Look at the last digit of the first ten numbers. What do you notice?

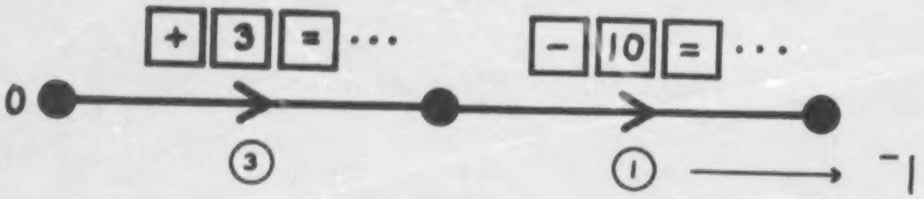
N 3

S: Each number 0 through 9 appears. The next ten numbers in the list will have 0 through 9 as their last digit, and the same for the next ten numbers, and so on. By using $\boxed{-}\boxed{10}\boxed{=}\ldots$, we can get any positive integer.

T: Very good. Now we know we can get any positive integer. What about the negative integers? Could Paf be $^{-}1$?

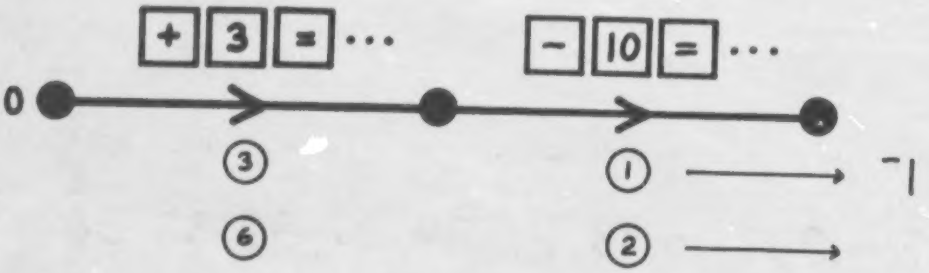
S: Yes; press $\boxed{+}\boxed{3}$ and $\boxed{=}$ three times followed by $\boxed{-}\boxed{10}$ and $\boxed{=}$ once.

Record this information below the picture on the board.



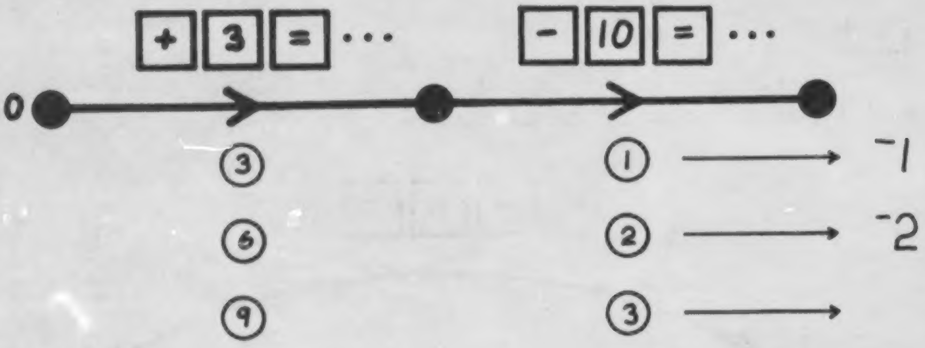
T: What number do I get when I double the number of times I press $\boxed{=}$ for each of the relations?

Add this information below the picture.



S: You get $^{-}2$, because $6 \times 3 = 18$ and $18 - 20 = {}^{-}2$.

T: And when I triple the number of times I press $\boxed{=}$?



S: You get $^{-}3$, because $9 \times 3 = 27$ and $27 - 30 = {}^{-}3$.

T: Could you get $^{-}4$, $^{-}5$, and $^{-}6$?

S: Just multiply the number of times you press $\boxed{=}$ by 4, 5, and 6 (respectively).

T: Do you think Paf could be any negative integer?

S: Yes, just continue multiplying the number of times you press $\boxed{=}$.

T: Very good. Could Paf be 0?

S: No, the dot for Paf is different from the dot for 0.

T: So, Paf could be any integer except 0.

Second clue

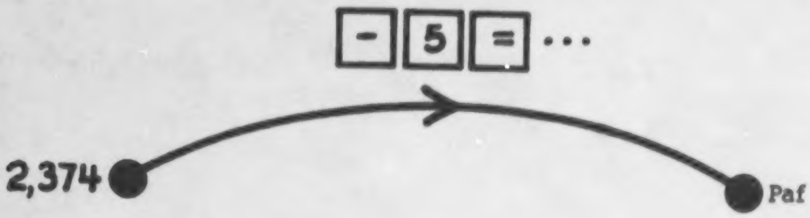
T: Paf is a positive divisor of 18. Can you give me one positive divisor of 18?

S: 3, because $3 \times 6 = 18$.

Continue until the class concludes that Paf could be 1, 2, 3, 6, 9, or 18.

Third clue

Draw this arrow picture on the board.



T: The third clue is given by this arrow picture. If you put 2,374 on the display of a hand-calculator and press $\boxed{-}\boxed{5}\boxed{=}$... as many times as you like, Paf will appear on the display. Who is Paf?

Let the students have a few minutes to explore the clue.

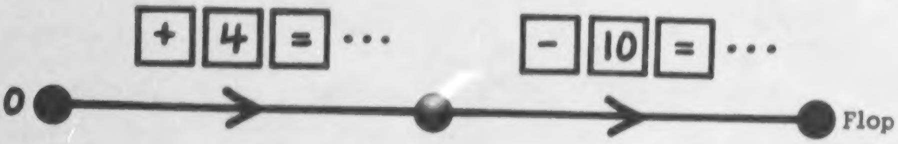
S: The last digit of the numbers that appear on the display is 4 or 9. Paf must be 9.

The class should conclude that Paf is 9.

Worksheets N15 and N16 are available.

When most of the class has completed N15, you may want to discuss this worksheet collectively. The discussion might develop in a manner similar to that of the detective story. For example, begin as follows.

Draw this arrow picture on the board.



T: What are some numbers that Flop could be?

Let students suggest and occasionally explain a number that Flop could be. List their suggestions on the board and encourage them to find the numbers in order. NOTE: Flop could be any even integer except 0. After you have a list of eight or ten numbers, begin to discuss the possibilities.

T: These are all even integers. Could Flop be an odd number?

S: No. We start with an even number and always add or subtract an even number.

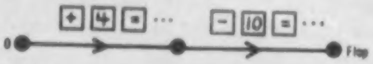
Accept any reasonable explanation, even if it is not the same as this one.

Discuss the second and third clues, in the usual way. The class should conclude that Flop is 4.

Name ____________________

N 15 *

First clue



Flop could be $^{-}6$, $^{-}4$, $^{-}2$, 2, 4, 6, 8, 10, 20, 40, 100, 102, 352, 400, 510, and so on.

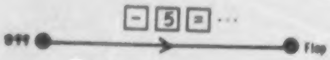
What do you notice about Flop? Flop could be any even integer except 0.

Second clue

Flop is a positive divisor of 12.

Flop could be 2, 4, 6, or 12.

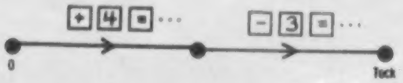
Third clue



Flop is 4.

Name ____________________

N 16 **



Could Tock be 1? yes 2? yes 3? yes 4? yes 5? yes 6? yes 7? yes 8? yes 9? yes 10? yes

Is there any whole number that Tock cannot be? yes; 0

Could Tock be $^{-}1$? yes $^{-}2$? yes $^{-}3$? yes $^{-}4$? yes $^{-}5$ yes $^{-}6$? yes

What do you notice about Tock? Tock could be any integer except 0.

2.2 Who is Crick?

CAPSULE LESSON SUMMARY

Solve a detective story in which clues involve patterns generated by combined use of the hand-calculator relations $\boxed{+}\boxed{3}\boxed{=}$ . . . and $\boxed{+}\boxed{7}\boxed{=}$ . . . and combined use of $\boxed{+}\boxed{2}\boxed{=}$ . . . and $\boxed{+}\boxed{5}\boxed{=}$ . . . , and in which the final clue is a three-string picture.

MATERIALS

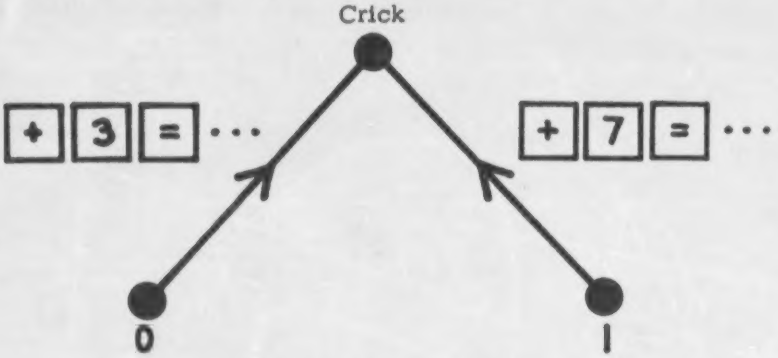
Teacher: Colored chalk
Student: Hand-calculator; pencil; Worksheet N17, N18

DESCRIPTION OF LESSON

Tell the class that they are going to solve a detective story about Crick.

First clue

Draw this arrow picture on the board and distribute hand-calculators.



T: Today we have a detective story about a secret number called "Crick". Crick is in this arrow picture. Which numbers could Crick be?

Let the students have a few minutes to explore this situation.

S: Crick could be 36.

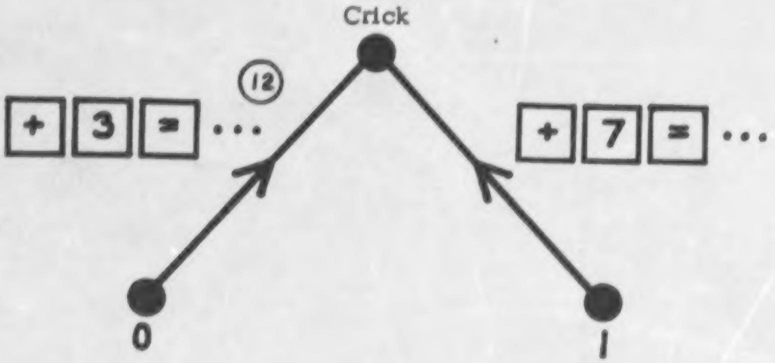
As the students find numbers that Crick could be, list them on the board. Occasionally, ask the class to verify that Crick could be a particular number. For example,

N 10

T: Crick could be 36. If we start from 0 and press $\boxed{+}\,\boxed{3}$, how many times must $\boxed{=}$ be pressed?

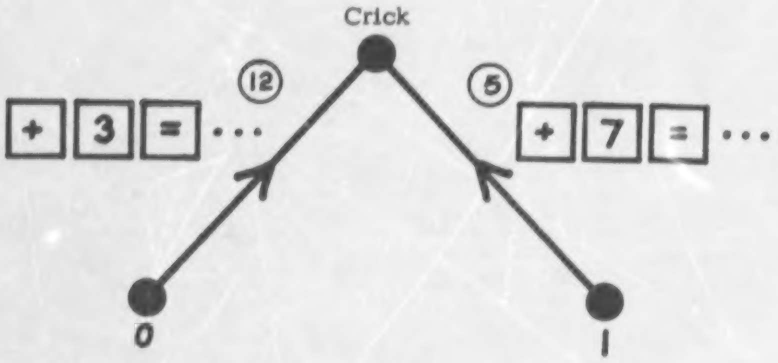
S: Twelve times, because 12 × 3 = 36.

Record this information on the board.



T: And if we start from 1 and press $\boxed{+}\,\boxed{7}$, How many times must $\boxed{=}$ be pressed?

S: Five times, because 1 + (5 × 7) = 36.



T: Which is the smallest number that Crick could be?

S: 15; 5 × 3 = 15 and 1 + (2 × 7) = 15.

T: Which other numbers could Crick be?

S: 78.

S: 246.

S: 57.

List the numbers that Crick could be in order even though your students may not suggest them in order.

15, 36, 57, 78, 99, ,162, ,246

After you have six to ten entries, ask:

T: Do you notice anything interesting about this list of numbers?

Give the students a few minutes to study the list.

S: There is a +21 pattern; 15 + 21 = 36, 36 + 21 = 57, and so on.

T: Why do you think there is a +21 pattern?

S: 21 is a multiple of 3 and a multiple of 7.

Accept any reasonable explanation.

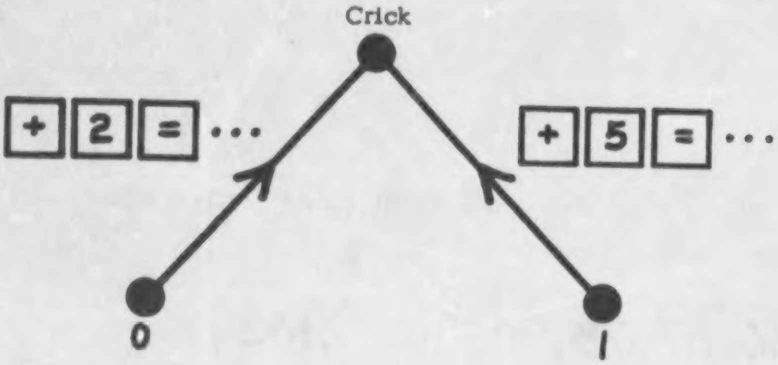
At this point you might ask the students to find several more numbers that Crick could be. The students should conclude that Crick could be:

15, 36, 57, 78, 99, 120, 141, 162, 183, 204, 225, 246, 267, and so on.

N 10

Second clue

Draw this arrow picture on the board.



T: Crick is in this arrow picture, too.
What does this picture tell us about Crick?

Let the students have a few minutes to explore the clue.

S: Crick must end with 6.

S: Yes, I found 6, 16, 26, 36, and so on.

T: Good. Now, use also the first clue. Which numbers could Crick be?

S: 36.

S: 246.

T: Which is the next number that Crick could be? How do you know?

S: 456 is the next number because 246 + 210 = 456.

T: That's correct, but why did you add 210?

S: In the first clue there is a +21 pattern and in the second clue there is a +10 pattern. So I started from 36 and picked every tenth number in the list (36, 246, 456,...). Picking every tenth number in the list is the same as adding 210 (10 × 21 = 210).

T: Good. What are some other numbers that Crick could be?

S: 666, because 456 + 210 = 666.

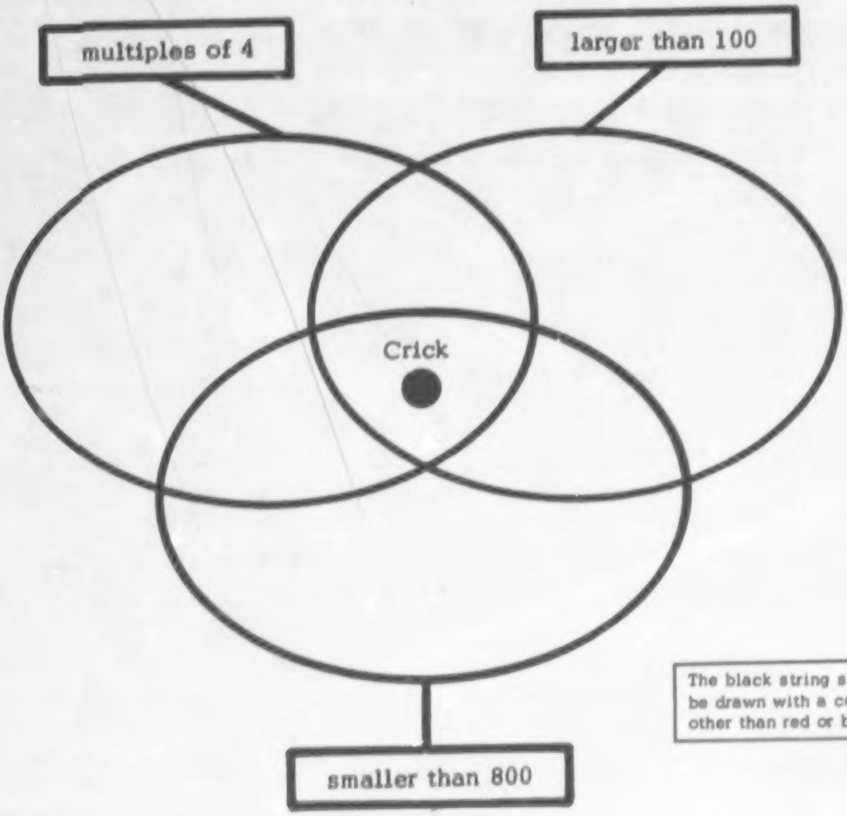
S: 876, because 666 + 210 = 876.

The students should conclude that Crick could be:

36, 246, 456, 666, 876, and so on.

Third clue

Draw this string picture on the board.



The black string should be drawn with a color other than red or blue.

N 10

T: The third clue is given by this string picture. What does this picture tell us about Crick?

S: Crick is a multiple of 4 and between 100 and 800.

T: Who is Crick?

Let the students have a few minutes to study the clue.

S: Crick could be 246 or 456 or 666; because these numbers are between 100 and 800.

T: Yes, but remember we said Crick is also a multiple of 4. Who is Crick?

S: Crick cannot be 246; 240 is a multiple of 4 (60 x 4 = 240). Next comes 244 followed by 248 and 246 is skipped.

S: Crick cannot be 666; 600 is a multiple of 4 and so is 60, so 660 must be a multiple of 4. Next comes 664 followed by 668 and 666 is skipped.

S: Crick is 456.

T: How many times 4 is 456?

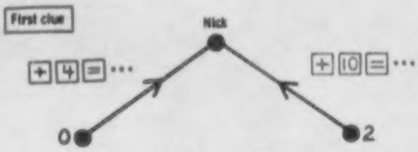
S: 114 x 4 = 456.

The class should conclude that Crick is 456. Worksheets N17 and N18 are available.

Name ____________________

N 17 *

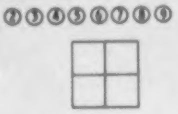
First clue



Nick could be 12, 32, 52, 72, 92, 112, and so on.

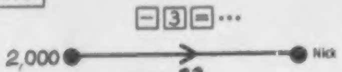
Second clue

Nick can be put on this Minicomputer board using exactly one of these checkers:



Nick could be 12, 32, or 72.

Third clue

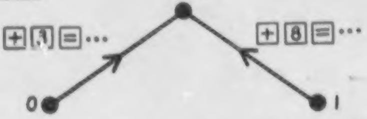


Nick is 32.

Name ____________________

N 18 **

First clue



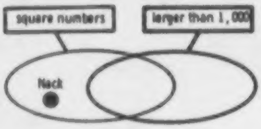
Nack could be 9, 33, 57, 81, 105, 129, 153, 177, 201, 225, 249, and so on.

Second clue

Nack is a multiple of 5.

Nack could be 105, 225, 345, 465, 585, and so on.

Third clue


square numbers
larger than 1,000
Nack

Nack is 225.

N 19

2.3 Greatest Common Divisor and Least Common Multiple Relations #1

CAPSULE LESSON SUMMARY

Play Guess My Rule games using greatest common divisor and least common multiple as the secret operations. That is, the rule for one game is: a * b is the greatest common divisor of a and b. For another game, the rule is: a * b is the least common multiple of a and b.

MATERIALS

Teacher:	Colored chalk
Student:	Paper; pencil

DESCRIPTION OF LESSON

NOTE: This lesson consists of four exercises. Depending on your class, the lesson might be too long for one period. Do not feel rushed to complete all four exercises in one period. It would be better to take your time and stop after Exercise 3 than to push the class beyond their abilities.

Exercise 1

Write this information on the board.

$$8 * 12 = 4 = 12 * 8$$

$$1 * 12 = 1 = 12 * 1$$

$$12 * 18 = \square = 18 * 12$$

T: I am thinking of a secret rule. I will give you some hints and you should try to guess my rule.

NOTE: The rule is: a * b is the greatest common divisor of a and b; for example, 8 * 12 = 4 because 4 is the greatest common divisor of 8 and 12. Do not tell the class the rule at this time.

T: Which number do you think belongs in this box?

Walk around the room and let students whisper their guesses in your ear. Acknowledge, aloud, any correct guess and deny any incorrect one. For example, "Yes, that's correct" or "No, 4 does not belong in the box." After a while invite a student to announce that 6 goes in the box or, if necessary, do so yourself. Continue the activity with more sentences involving * , letting students provide the numbers in boxes.

$$12 * 20 = \boxed{4} = 20 * 12$$

$$12 * 21 = \boxed{3} = 21 * 12$$

$$14 * 21 = \boxed{7} = 21 * 14$$

$$80 * 100 = \boxed{20} = 100 * 80$$

$$8 * 8 = \boxed{8}$$

$$24 * 30 = \boxed{6} = 30 * 24$$

If there are a few students who have not discovered the rule yet, ask some leading questions.

T: Do you notice anything interesting about these numbers?

S: The secret number is a divisor of each of the other two numbers.

T: Good. That is part of my secret rule. What else do you notice?

S: The secret number is the largest number which divides both the other two numbers.

N 19

If this observation is not forthcoming, discuss one of the sentences; for example,

$$12 * 18 = 6$$

T: Name the positive divisors of 12.

As the students name the positive divisors of 12, list them as illustrated below.

12	*	18	=	6
1				
2				
3				
4				
6				
12				

T: Name the positive divisors of 18.

List the positive divisors of 18 as the students name them.

12	*	18	=	6
1		1		
2		2		
3		3		
4		6		
6		9		
12		18		

T: Which numbers are in both lists?

S: 1, 2, 3, and 6.

T: These numbers are the common divisors of 12 and 18. Which of these common divisors is the largest?

S: 6.

Discuss one or two other sentences and be sure the class knows your secret rule.

Ask the students to solve the problems written below. The answers are in the boxes. Encourage students who have difficulty to list the positive divisors of each number as suggested in the previous dialogue.

$$9 * 24 = \boxed{3}$$

$$12 * 40 = \boxed{4}$$

$$75 * 100 = \boxed{25}$$

$$(12 * 30) * 8 = \boxed{2}$$

$$12 * (30 * 8) = \boxed{2}$$

Be sure to check the problems collectively. You may wish to discuss the solutions for the last two problems.

T (pointing to "(12 * 30) * 8 = ☐ "): What do we do first?

S: Calculate "12 * 30", the largest number which is a divisor of both 12 and 30.

T: What number is 12 * 30?

N 19

S: 6.

Write "6" on the board above "(12 * 30)".

$$(12 \overset{6}{*} 30) * 8 = \square$$

T: Now, what do we do?

S: Find the largest number which is a divisor of both 6 and 8.

T: What number is 6 * 8?

S: 2.

Write "2" in the box.

$$(12 \overset{6}{*} 30) * 8 = \boxed{2}$$

Discuss "12 * (30 * 8) = $\square$" in a similar manner.

T: Do you notice anything interesting about these last two problems?

S: 2 is the largest (or greatest) common divisor of all three of the numbers.

S: It makes no difference where the parentheses are written.

Accept any reasonable comments.

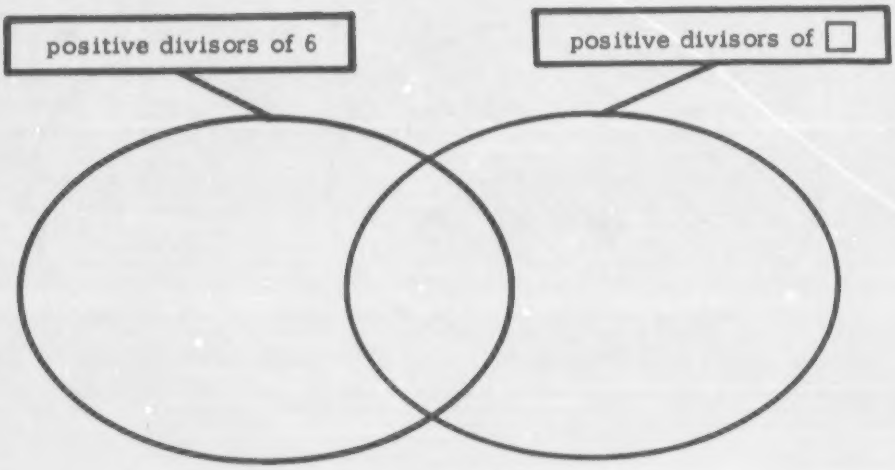
Exercise 2

Write this information on the board.

$$6 * \square = 2$$

T: Try to find as many numbers as you can that $\square$ could be. Try to find a pattern for the numbers that you find.

Let the students work independently for a few minutes while you draw this string picture on the board.



T: Name the positive divisors of 6.

S: 1, 2, 3, and 6.

T: Put the positive divisors of 6 in the string picture.

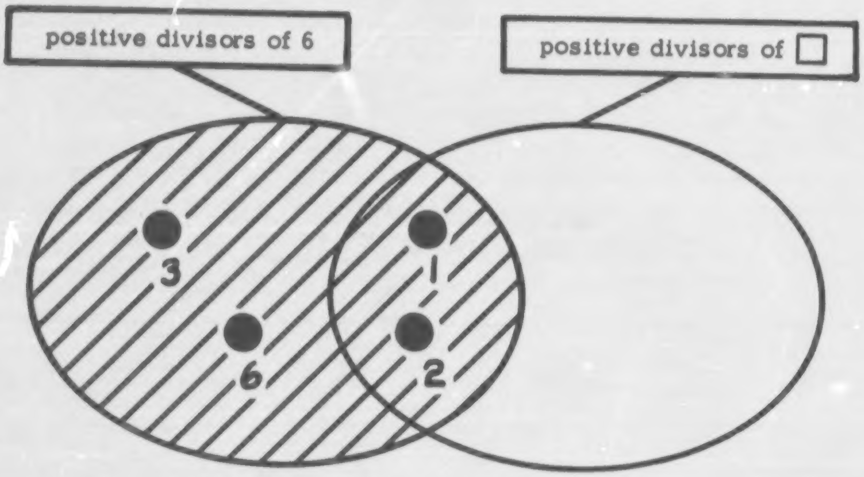
Perhaps there will be a few false starts, but the students should catch mistakes and correct themselves.

S: 1 is in the middle, because it is a positive divisor of both numbers no matter what $\square$ is.

S: 2 is the only other number that can be in the middle, because 2 is a positive divisor of 6 and of □ and we know that 2 is the largest common divisor.

S: 3 and 6 must be inside the red string but outside the blue string.

Be sure to ask the students to put the numbers in the string picture as they discuss them. At the conclusion of this part of the discussion, someone might suggest that part of the picture be hatched as shown below.



T: What can you tell me about □ ?

S: □ is even, because 2 is a divisor of □ .

S: □ is neither a multiple of 6 nor of 3, because 3 and 6 are not divisors of □ .

S: □ is any even number which is not a multiple of 3.

T: Which numbers could □ be?

Let the students give several examples such as:

2, 4, 8, 10, 14, 16, 20, 22, ...

Exercise 3

Write this information on the board.

□ * △ = 6

□	△

T: Copy the chart and try to find at least ten pairs of numbers for □ and △.

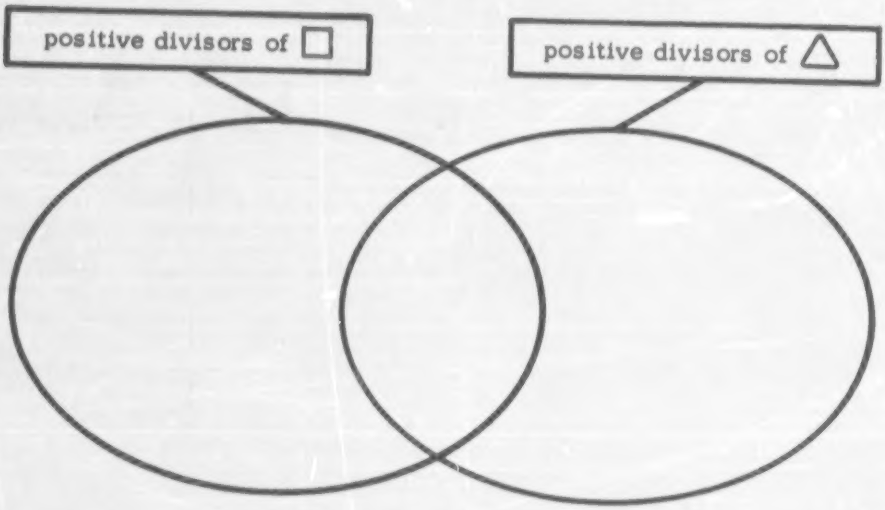
Let the students work independently for several minutes. When you observe a student having difficulty, suggest that the student choose some multiple of 6 for □ and then try to find some number that △ could be.

The students' charts should begin to look similar to this one with many different entries.

□	△
18	30
12	18
30	36
6	6
24	30
54	12
12	54
36	6
6	36

N 19

After most student appear to have a method for finding pairs of numbers for □ and △, draw this string picture on the board.



T: Which numbers can we put in this string picture for sure?

Let the students consider the situation for a few minutes.

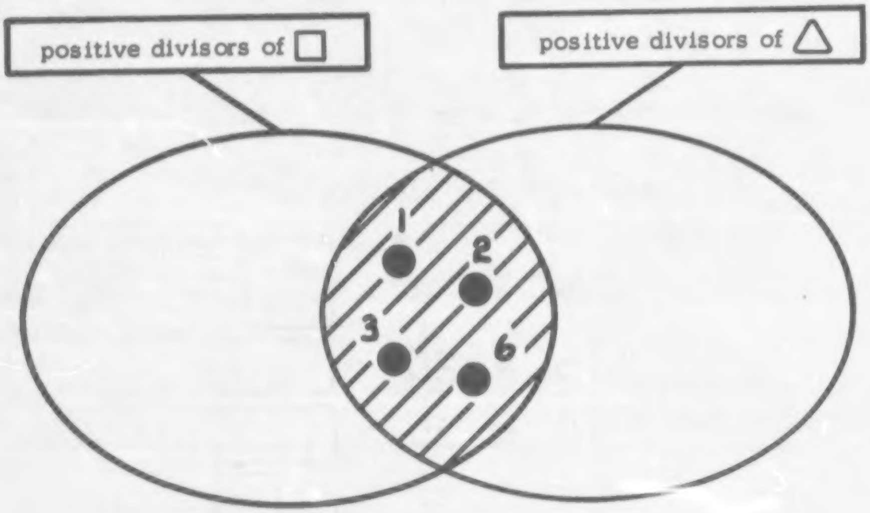
S: 6 can go in the middle because 6 is the greatest common divisor of □ and △.

S: 1 can go in the middle because 1 is a divisor of every number.

S: If 6 is a divisor of both □ and △, then so also are 2 and 3. Both 2 and 3 are in the middle.

Ask students to put numbers in the string picture as they are discussed.

S: The middle should be hatched now.



Choose a few examples (pairs of numbers for □ and △) from the students' charts and show that this picture "works."

Exercise 4

NOTE: Because of time constraints, you may prefer to omit this exercise now. However, present it to your class (on an adjustment day, for example) prior to Lesson 2.5 Greatest Common Divisor and Least Common Multiple Relations #2.

Write this information on the board.

$$4 * 6 = 12 = 6 * 4$$

$$3 * 6 = 6 = 6 * 3$$

$$8 * 12 = \square = 12 * 8$$

T: I am thinking of a new secret rule. I have given you some hints; try to guess my rule.

N 19

NOTE: The rule is: a * b is the least common multiple of a and b; for example, 4 * 6 = 12 because 12 is the least common multiple of 4 and 6. Do not tell the students the rule at this time.

T: Which number is this (point to "☐")? (24)

Walk around the room and let students whisper their guesses in your ear. Continue the activity with the problems below (the answers are in the boxes).

$$8 * 8 = \boxed{8}$$
$$15 * 20 = \boxed{60}$$
$$40 * 30 = \boxed{120}$$
$$24 * 36 = \boxed{72}$$
$$50 * 60 = \boxed{300}$$

By now several students should have discovered the rule. Discuss it collectively in a manner similar to that of Exercise 1. Some students will benefit by choosing a problem and listing some of the multiples of each of the numbers to emphasize the notion of common multiple. For example,

$8 * 12 = 24$

8	12
16	24
24	36
32	48
40	60
48	

After common multiples have been observed, pick out the smallest (or least) common multiple.

For the remainder of the lesson ask the students to copy and solve these problems. Be sure to check the problems collectively. The answers are in boxes.

$10 * 12 = \boxed{60}$

$10 * 15 = \boxed{30}$

$12 * 36 = \boxed{72}$

$24 * 30 = \boxed{120}$

$60 * 70 = \boxed{420}$

N 25

2.4 Who is Clip?

CAPSULE LESSON SUMMARY

Examine characteristics of the sum of two multiples of 3, the sum of a multiple of 3 and a non-multiple of 3, and the sum of two non-multiples of 3. Solve a detective story in which the clues involve multiples of 3, patterns generated by moving one checker in a configuration on the Minicomputer, numbers which can be put on the ones' board of the Minicomputer with weighted checkers, and a string picture.

MATERIALS

Teacher: Colored chalk

Student: Pencil; Worksheets N19, N20, N21, N22

DESCRIPTION OF LESSON

Exercise 1

Write this expression on the board.

$$\square + \triangle$$

T: Nabu put a multiple of 3 in the square and a multiple of 3 in the triangle. Then he added the two numbers. What can you tell me about the sum?

Let the students have a few minutes to consider the situation.

S: The sum is a multiple of 3. A multiple of 3 plus a multiple of 3 is a multiple of 3.

T: Let's try some examples. Give us a multiple of 3 for $\square$.

S: 9.

T: Give us a multiple of 3 for △.

S: 36.

T: What number is 9 + 36?

S: 45.

T: Is 45 a multiple of 3?

S: Yes.

Illustrate this fact with several more examples.

Repeat this activity with other characteristics for □ and △. Two situations are outlined below. Be sure to illustrate these facts with several examples.

□ : Multiple of 3

△ : Non-multiple of 3

Sum : Non-multiple of 3

Examples

6 + 2 = 8
12 + 7 = 19
21 + 11 = 32
18 + 8 = 26
24 + 13 = 37

□ : Non-multiple of 3

△ : Non-multiple of 3

Sum : The sum can be either a multiple of 3 or a non-multiple of 3 depending on which non-multiples of 3 are chosen.

Examples

5 + 1 = 6 (Multiple of 3)
5 + 2 = 7 (Non-multiple of 3)
16 + 22 = 38 (Non-multiple of 3)
43 + 35 = 78 (Multiple of 3)
17 + 26 = 43 (Non-multiple of 3)
25 + 38 = 63 (Multiple of 3)

N 25

Exercise 2

Present this detective story about a secret number named "Clip".

First clue

T: Clip is a multiple of 3. Name some multiples of 3.

Let the students name several multiples of 3.

T: On you paper, write the smallest number larger than 1,000 that Clip could be. Which number is it?

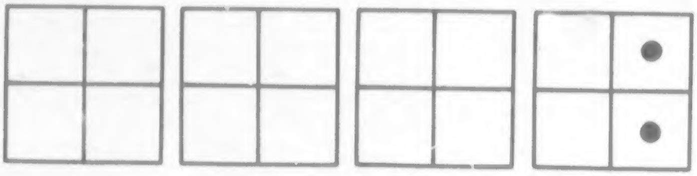
S: 1,002; because 999 is a multiple of 3 so 999 + 3 = 1,002 is the next larger multiple of 3.

T: Which is the largest number smaller than 2,000 that Clip could be?

S: 1,998, because 1,002 + 999 = 2,001 is a multiple of 3 and 2,001 - 3 = 1,998.

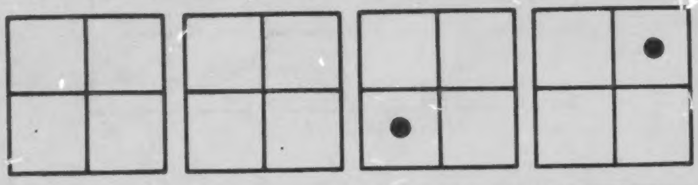
Second clue

Display four Minicomputer boards with this configuration of checkers.



T: If you move one of these checkers to another square, you will get Clip. You may use more than just these four Minicomputer boards. Which numbers could Clip be?

Invite students to move one checker on the Minicomputer and find possibilities for Clip. When necessary, remind a student that Clip is a multiple of 3. Some numbers that Clip could be are shown below.



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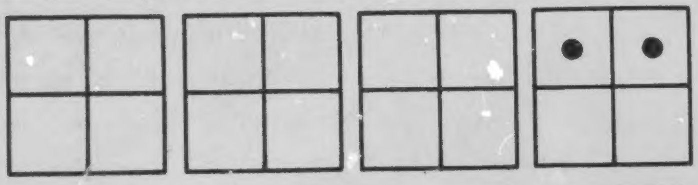

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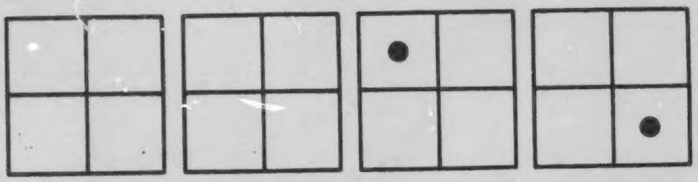

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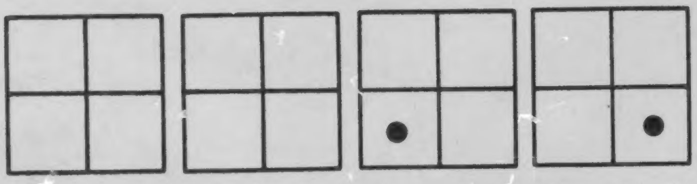


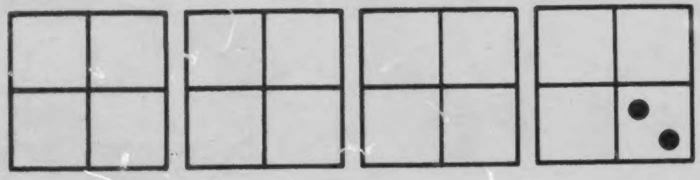
= 21

When it is obvious that most students understand the clue, continue as follows.

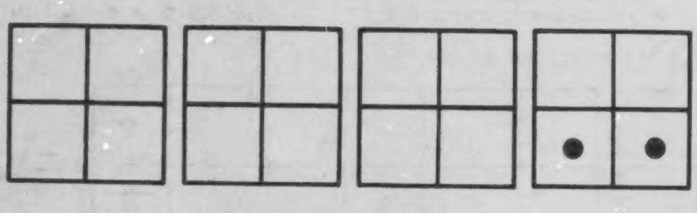
T: Let's try to be more systematic about finding numbers that Clip could be. Perhaps we will observe a pattern.

I will move the checker on the 4-square to other squares. Each time the number is a multiple of 3, raise your hand.

Start by moving the checker to the 1-square...



then to the 2-square...



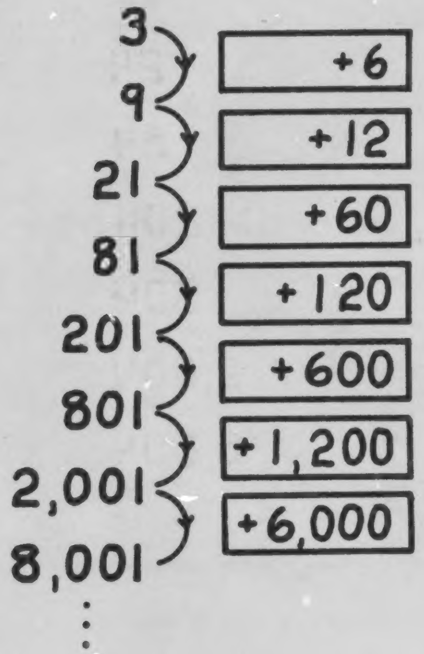
Since 3 is a multiple of 3, the students should raise their hands. Write "3" on the board to begin a list of possibilities for Clip. Continue in this manner moving the checker from square to square. After moving this checker to all the other squares (on the four displayed boards), your list of numbers for Clip should have at least eight entries.

3
9
21
81
201
801
2,001
8,001
⋮

S: It always works when the checker is moved to a red square or a brown square.

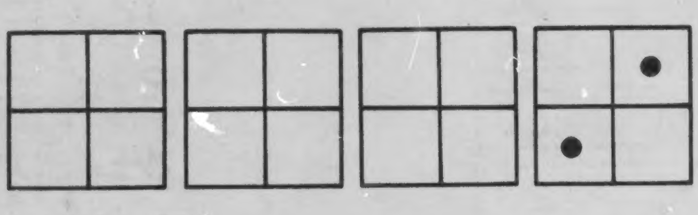
98

Add these red arrows to your list and ask the students to label them. The answers are in the boxes.



T: Now, I will move the checker from the 1-square. Again, raise your hand when we have a multiple of 3.

Start by moving the checker to the 2-square.



Since 6 is a multiple of 3, the students should raise their hands. Write "6" on the board to begin a second list of possibilities for Clip. Continue moving the checker from square to square. After moving this checker to all the other squares (on the four displayed boards), your second list should also have at least eight numbers that Clip could be.

6

12

24

84

204

804

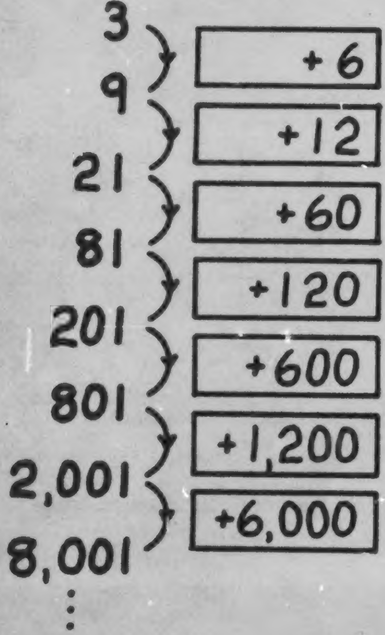
2,004

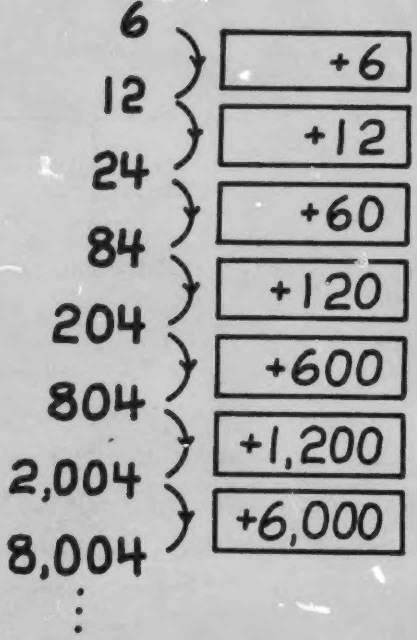
8,004

⋮

S: Again it works when the checker is moved to a red square or a brown square.

Draw red arrows in the same manner as for the first list and ask the students to label them. For comparison, both lists are shown below. The answers are in boxes.


3
9
21
81
201
801
2,001
8,001
+6
+12
+60
+120
+600
+1,200
+6,000

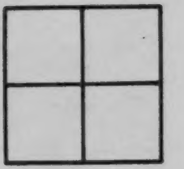

6
12
24
84
204
804
2,004
8,004
+6
+12
+60
+120
+600
+1,200
+6,000

The class should conclude that Clip could be any of the numbers in the two lists.

Third clue

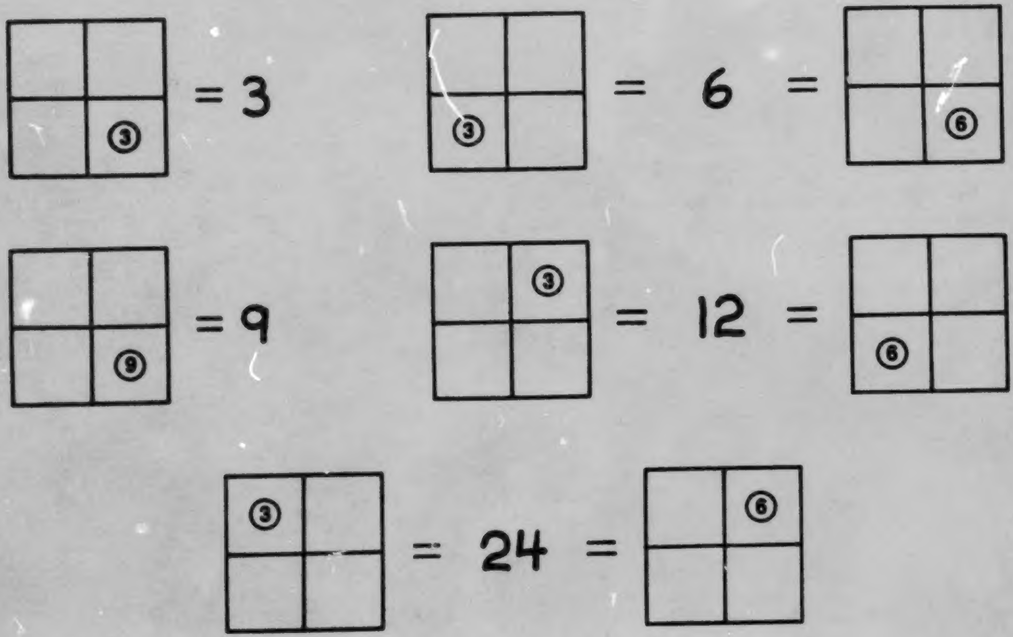
Display one Minicomputer board and this set of weighted checkers.

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



T : Clip can be put on this ones' board of the Minicomputer using exactly one of these checkers.

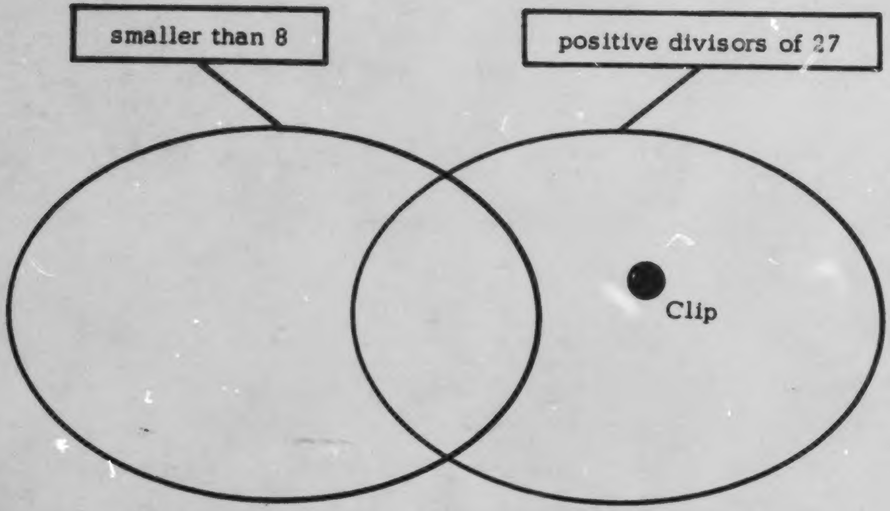
Ask students to place one checker on the Minicomputer to show possibilities for Clip. The class should conclude that Clip is one of the following numbers:



N 25

Fourth clue

Draw this string picture on the board.


smaller than 8
positive divisors of 27
Clip

T: Clip is in this string picture. Who is Clip?

Let the students study the string picture for a moment.

S: Clip is a positive divisor of 27 and is not smaller than 8, so Clip must be 9.

The students should conclude that Clip is 9.

Worksheets N19, N20, N21, and N22 are available.

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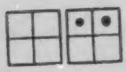
Name ____________

N 19 *

Add a regular checker to the Minicomputer to get a multiple of 3.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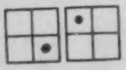
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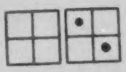
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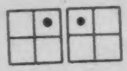
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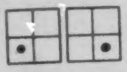
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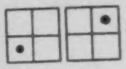
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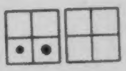
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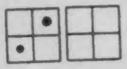
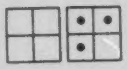
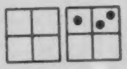

= 60

Name ____________

N 20 **

By moving exactly one of these checkers to another square, find as many multiples of 4 as you can.

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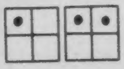
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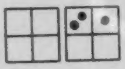
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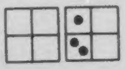
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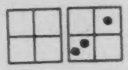
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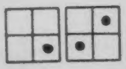
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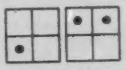
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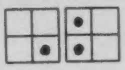
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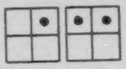
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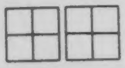
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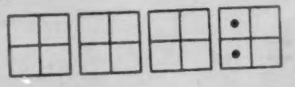
Name ____________

N 21 ***

First clue



Nick is a multiple of 4 and can be put on these Minicomputer boards by moving exactly one of these checkers to another square.

Nick could be 4, 12, 16, 28, 48, 88, 108, 208, 408, 808, 1,008, 2,008, 4,008, or 8,008

Second clue

Nick is a multiple of 6.

Nick could be 12, 48, 108, 408, 1,008, or 4,008

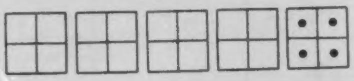
Third clue

Nick is a multiple of 9 larger than 200.

Nick is 1,008.

Name ____________

N 22 ****



Flip is a multiple of 9 and can be put on the Minicomputer by moving exactly one of these checkers to another square.

Flip could be 9, 18, 27, 54, 207, 414, 2,007, 4,014, 20,007 or 40,014.

Describe any interesting patterns that you notice.

One interesting pattern is that the sum of the digits of each of these numbers is a multiple of 9. Your students may observe other patterns.

2.5 Greatest Common Divisor and Least Common Multiple Relations #2

CAPSULE LESSON SUMMARY

Review what is meant by the greatest common divisor (gcd) and the least common multiple (lcm) of two numbers. Introduce the notation " $\sqcap$ " for gcd and " $\sqcup$ " for lcm. Explore relationships between statements about $\sqcap$ ($\sqcup$) and string pictures with strings for positive divisors (multiples). Present a detective story where the clues involve $\sqcup$, prime divisors, and a string picture where the strings are for positive divisors of certain numbers.

MATERIALS

Teacher: Colored chalk

Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Write this information and these problems on the board.

$\sqcap$: greatest common divisor	$\sqcup$: least common multiple
$12 \sqcap 18 = \square$	$12 \sqcup 18 = \square$
$15 \sqcap 20 = \square$	$15 \sqcup 20 = \square$
$1 \sqcap 9 = \square$	$1 \sqcup 9 = \square$
$8 \sqcap 9 = \square$	$8 \sqcup 9 = \square$
$14 \sqcap 21 = \square$	$14 \sqcup 21 = \square$

T: In a preceding lesson, we were introduced to the notion of the greatest common divisor and the least common multiple of two numbers. They were the secret rules for some Guess My Rule games, and each time we used "*" to denote the rule. For convenience and to avoid confusion, let's use the two symbols on the board instead of the "*" (star).

Solve the problems collectively, reviewing the meanings of "greatest common divisor" and "least common multiple".

$\sqcap$: greatest common divisor

$12 \sqcap 18 = \boxed{6}$

$15 \sqcap 20 = \boxed{5}$

$1 \sqcap 9 = \boxed{1}$

$8 \sqcap 9 = \boxed{1}$

$14 \sqcap 21 = \boxed{7}$

$\sqcup$: least common multiple

$12 \sqcup 18 = \boxed{36}$

$15 \sqcup 20 = \boxed{60}$

$1 \sqcup 9 = \boxed{9}$

$8 \sqcup 9 = \boxed{72}$

$14 \sqcup 21 = \boxed{42}$

Continue this activity by writing the problems below on the board. Answers are in the boxes ; do not write them on the board. Ask the students to copy the problems and solve them on their own. Check the individual work collectively.

$12 \sqcap 20 = \boxed{4}$

$15 \sqcap 25 = \boxed{5}$

$75 \sqcap 100 = \boxed{25}$

$(12 \sqcap 18) \sqcap 24 = \boxed{6}$

$12 \sqcap (18 \sqcap 24) = \boxed{6}$

$12 \sqcup 20 = \boxed{60}$

$15 \sqcup 25 = \boxed{75}$

$75 \sqcup 100 = \boxed{300}$

$(12 \sqcup 18) \sqcup 24 = \boxed{72}$

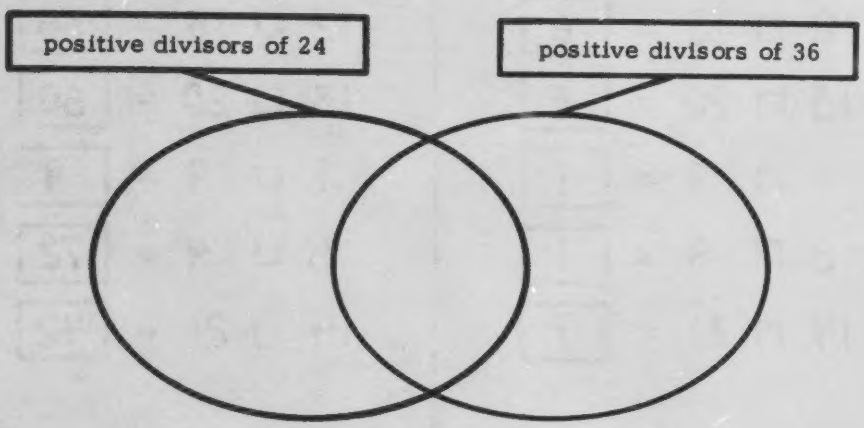
$12 \sqcup (18 \sqcup 24) = \boxed{72}$

N 28

Exercise 2

Write this information on the board.

$$24 \sqcap 36 = \square$$

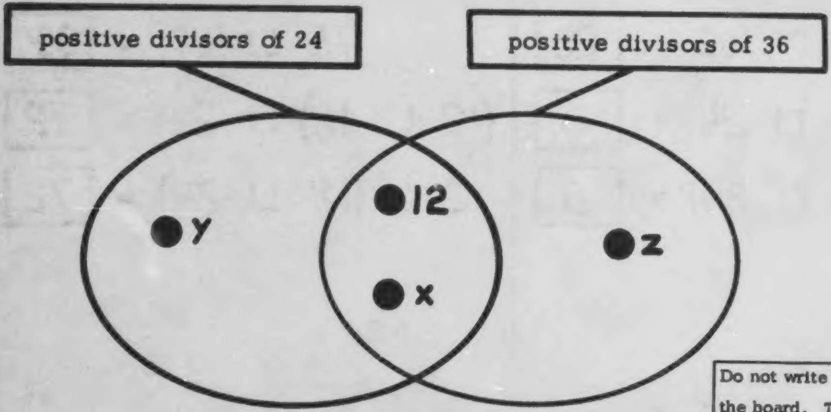

positive divisors of 24
positive divisors of 36

T: What number is $24 \sqcap 36$?

S: 12.

T: Good. Where is 12 in the string picture?

Invite a student to put 12 in the string picture. Then put another dot in the middle region (the dot labeled "x" in the following picture; dots labeled "y" and "z" will be added later).


positive divisors of 24
positive divisors of 36
12
y
x
z

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

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T (pointing to x): Which numbers could be here?

The class should conclude that x could be 1, 2, 3, 4, or 6.

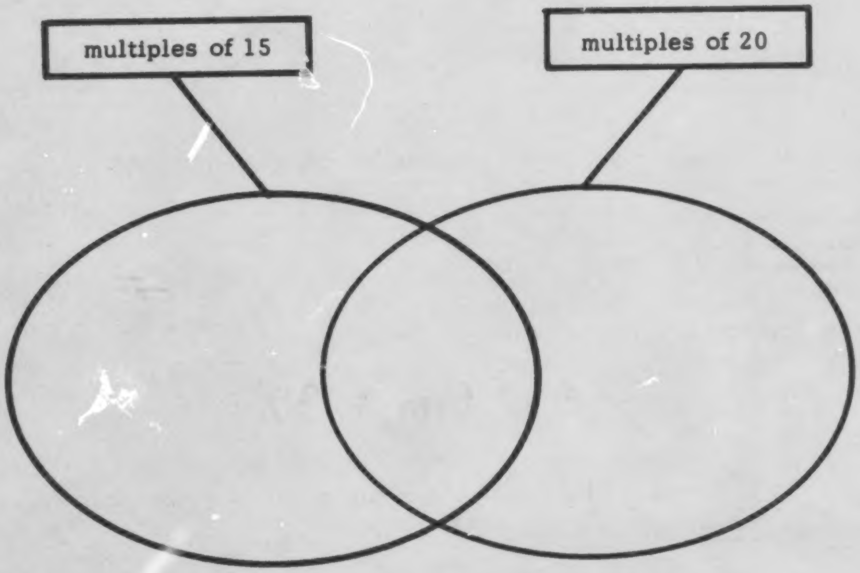
Proceed in a similar manner with dots in other regions (the dots labeled "y" and "z"). The class should conclude that y could be 8 or 24 and that z could be 9, 18, or 36.

T (pointing to the middle region): What do you notice about the numbers in this region?

S: Each number in the middle region is a positive divisor of 12.

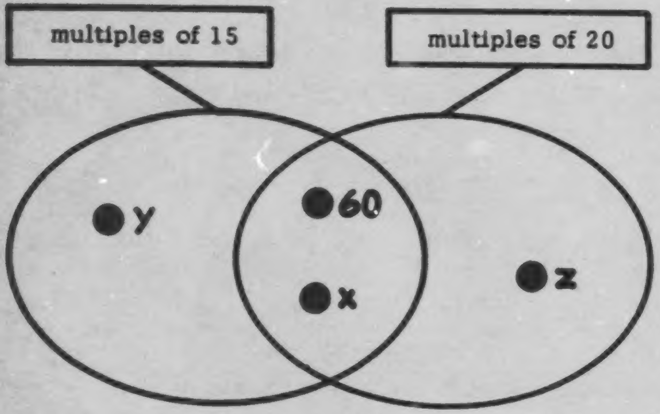
Repeat this exercise using the following situation.

$$15 \sqcup 20 = \square$$



The next illustration outlines the results that the class should discover about this situation.

$$15 \sqcup 20 = \boxed{60}$$



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

x could be: 60, 120, 180, 240, 300, 360,...
y could be: 15, 30, 45, 75, 90, 105, 135,...
z could be: 20, 40, 80, 100, 140, 160, 200,...

Exercise 3

Present this detective story about a secret number named "Bim".

First clue

Write this information on the board.

$$6 \sqcup \text{Bim} = 30$$

T: Bim is a secret whole number. What information does this clue give us about Bim?

S: 30 is the least common multiple of 6 and Bim.

S: Bim is a positive divisor of 30.

T: Yes. Write the positive divisors of 30 on your paper.

List the positive divisors of 30 on the board as the students give them to you.

1, 2, 3, 5, 6, 10, 15, 30

T: Which of these numbers cannot be Bim?

S: Bim cannot be 1, because 6 $\sqcup$ 1 = 6, not 30.

Cross off "1" from your list.

S: Bim cannot be 2, 3, or 6. The least common multiple of each of these numbers and 6 is 6, not 30.

T: Do you notice anything interesting about the numbers we crossed off?

S: Each number is a positive divisor of 6.

The students should conclude that Bim is 5, 10, 15, or 30.

Second clue

T: The second clue is Bim has only odd prime divisors. Which numbers could be Bim?

S: Bim cannot be 10, because 2 is a prime divisor of 10.

S: Cross off "30", because 30 has 2 for a prime divisor.

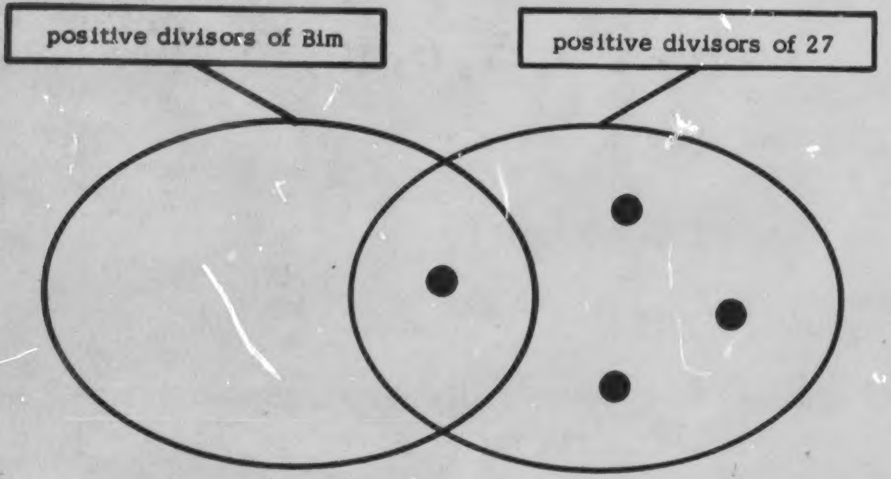
S: Bim could be 5 or 15. Their only prime divisors are 5 or 3.

N 28

The students should conclude that Bim is 5 or 15.

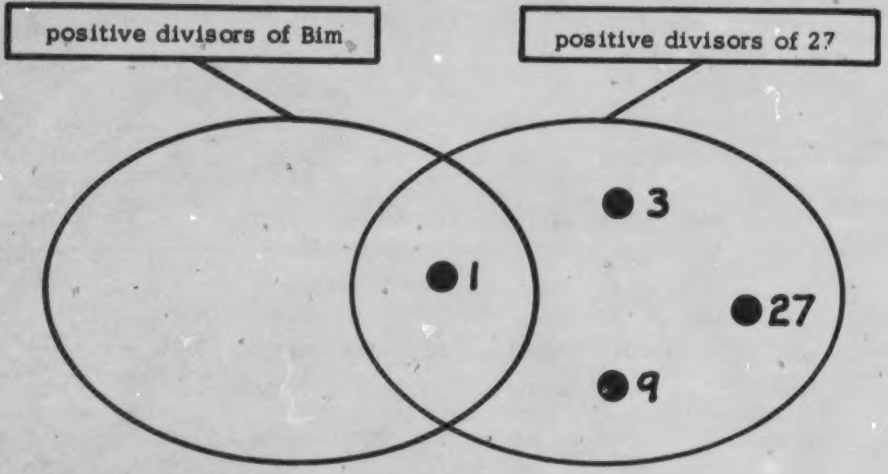
Third clue

Draw this string picture on the board.


positive divisors of Bim
positive divisors of 27

T: The third clue is given by this string picture. Who can label the dots?

Invite students to label the dots.


positive divisors of Bim
positive divisors of 27
1
3
27
9

T: What information does this picture give us about Bim?

S: 1 is a positive divisor of Bim; 3, 9, and 27 are not positive divisors of Bim.

T: Who is Bim?

S: Bim is 5, because 3 is a divisor of 15 but 3 is not a divisor of Bim.

The students should conclude that Bim is 5.

2.6 The Relation $=_3$

CAPSULE LESSON SUMMARY

Play a Guess My Rule game with a secret rule for arrows. The secret rule is: there is a arrow from x to 0 if x is a multiple of 3, from x to 1 if x is 1 more than a multiple of 3, and from x to 2 if x is 2 more than a multiple of 3. With the aid of a string picture, explore the possibilites when you add two numbers which are not multiples of 3. Use the string picture to introduce $=_3$ (read "equals modulo 3") and observe that two numbers are equal modulo 3 if and only if their difference is a multiple of 3.

MATERIALS

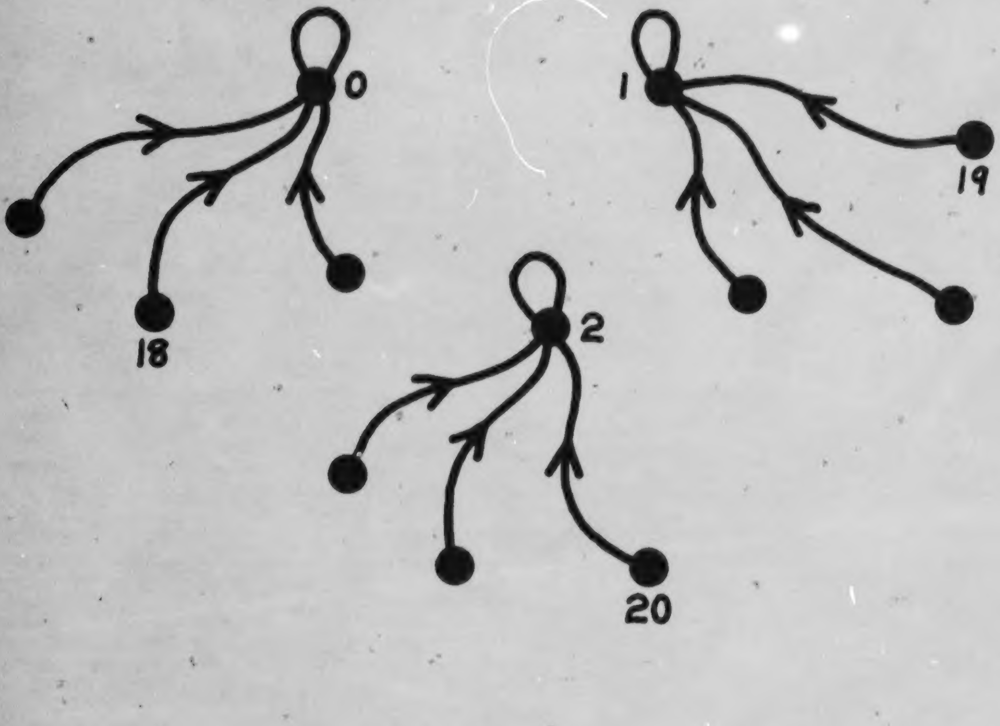
Teacher: Colored chalk

Student: Paper; pencil; Worksheets N23, N24, N25

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.

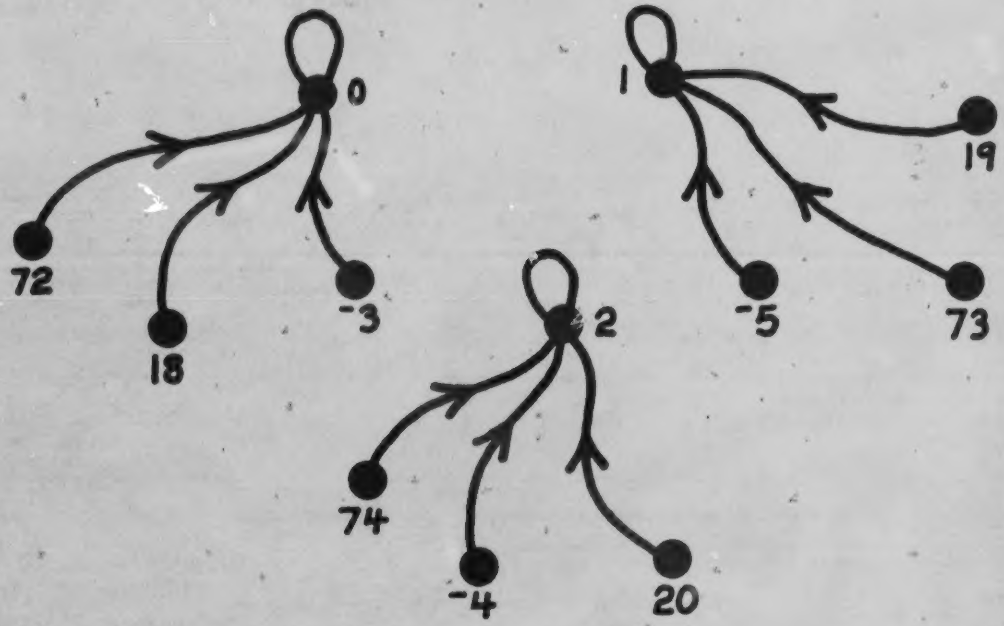


T: Can you guess my rule for the red arrows? There are several clues in this picture. Study the picture and in a moment maybe you will know how to label another dot. I'll tell you if you are correct.

NOTE: The rule for the red arrows is: each multiple of 3 is the starting number for an arrow ending at 0, each number which is 1 more than a multiple of 3 is the starting number for an arrow ending at 1, and each number which is 2 more than a multiple of 3 is the starting number for an arrow ending at 2. Do not tell the rule to the class at this time.

Allow several minutes for the class to study the clue. Invite students to label dots and each time tell the class whether or not a dot is correctly labeled (erase incorrect labels).

We cannot predict which numbers your students will choose, but your picture might look like this.



Continue this activity until it is obvious that some students know your rule. You may add more arrows to your picture if necessary.

T: Who can explain my rule?

S: Each multiple of 3 is the starting number of an arrow ending at 0. Each number which is 1 more than a multiple of 3 is the starting number of an arrow ending at 1. Each number which is 2 more than a multiple of 3 is the starting number of an arrow ending at 2.

Erase all the labels for the dots except "0", "1", and "2". Write this list of numbers on the board.

101, 88, 57, 47, 276, 163, 68, ⁻33, 91

NOTE: Feel free to adjust this list to the numerical level of your class; for example, if you are teaching a low ability class you may prefer to replace the negative integer.

T: Copy the arrow picture and put these numbers in your picture.

Observe the students' independent work, helping those who have difficulty getting started. Some leading questions may help students; for example,

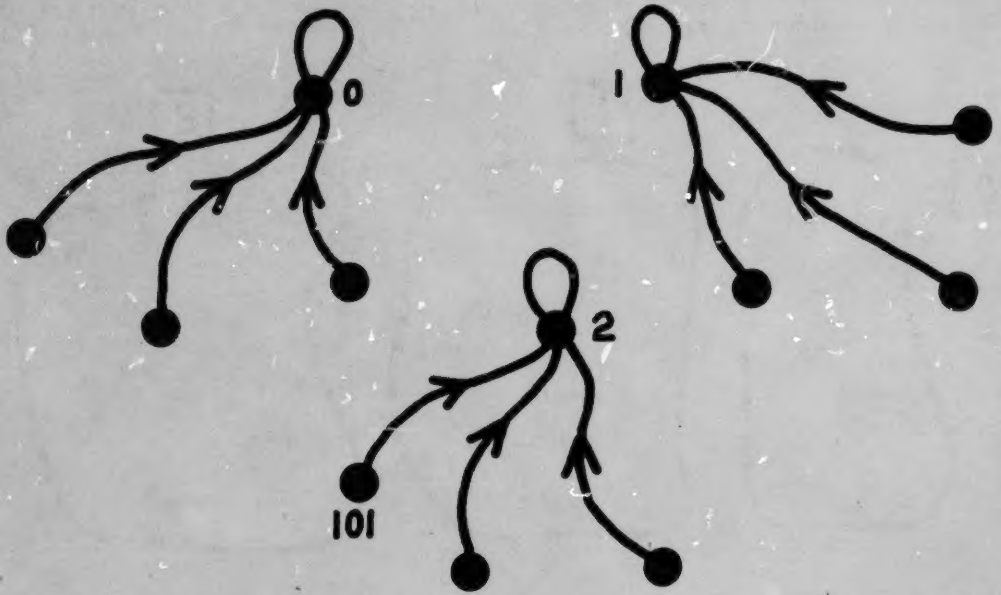
T: Is 101 a multiple of 3?

S: No, but 99 is a multiple of 3.

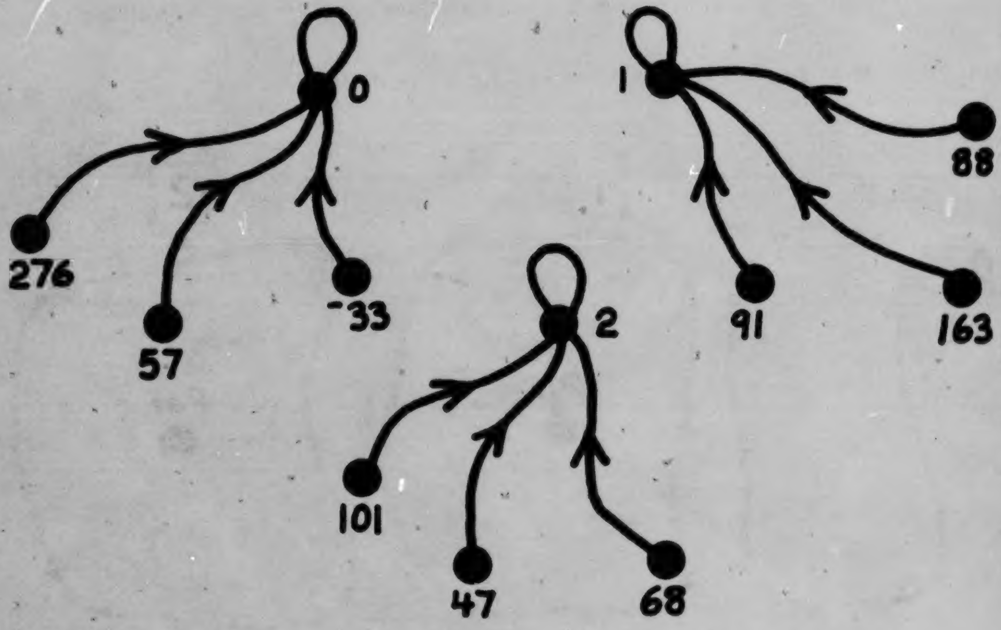
T: What can you say about 101 then?

S: 101 = 99 + 2, so 101 is 2 more than a multiple of 3.

T: Show us where 101 could be in the picture.



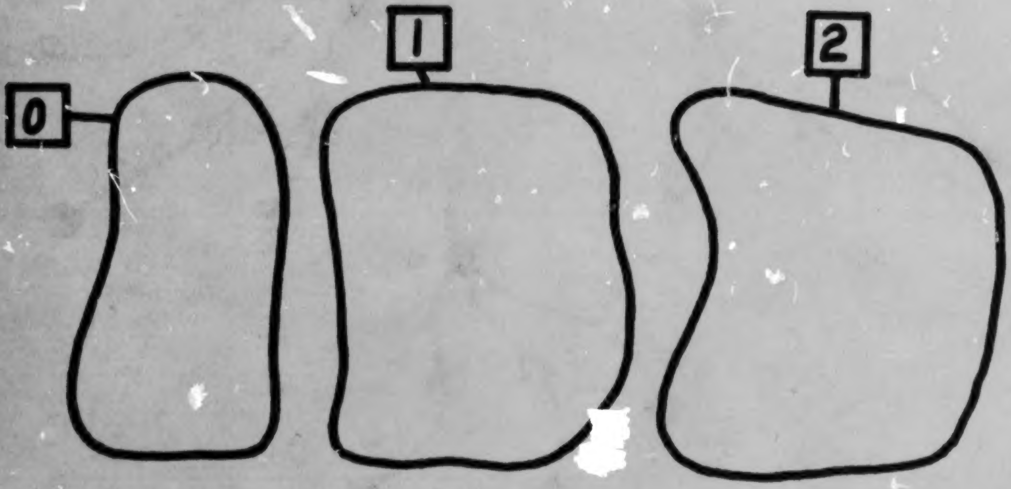
Check the independent work by collectively putting each number in the arrow picture.


0
1
2
276
57
⁻33
88
91
163
101
47
68

N 34

Exercise 2

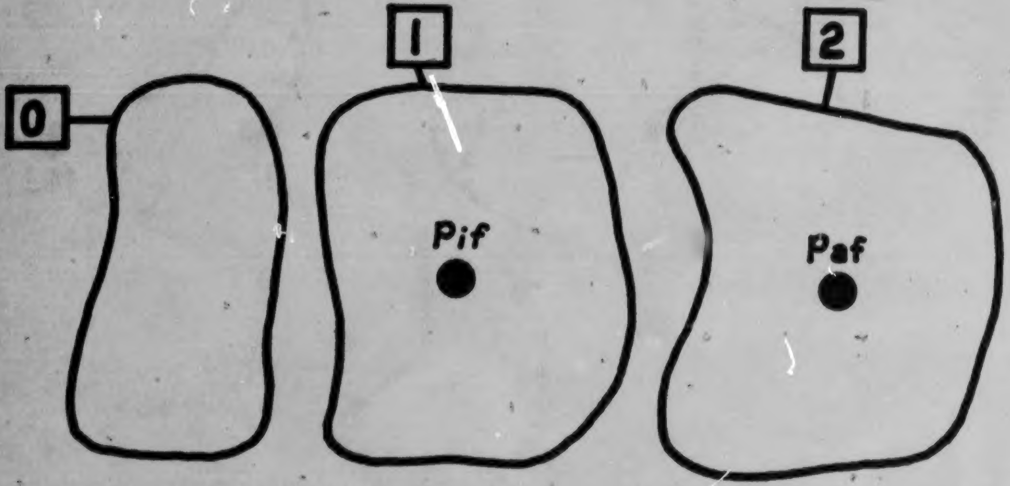
Draw these strings on the board.



T: The string labeled "0" is for the multiples of 3. What do you think the other two strings are for?

S: Numbers which are 1 more than multiples of 3 are in the 1-string, and numbers which are 2 more than multiples of 3 are in the 2-string.

Add "Pif" and "Paf" to your picture as illustrated below.



T: Choose any numbers you wish for Pif and for Paf, so long as Pif is in the 1-string and Paf is in the 2-string. In which string is Pif + Paf?

Let the students have a few minutes to explore this situation.

S: Pif + Paf is in the 0- string.

T: Can you explain why?

S: When you add a number which is 1 more than a multiple of 3 to a number which is 2 more than a multiple of 3, you get a number which is 3 more than a multiple of 3, or simply a multiple of 3.

Accept any reasonable explanation even if it is not exactly like the above.

T: Give me a number for Pif.

S: 10.

T: And a number for Paf.

S: $^-16$.

T: What do you notice about Pif + Paf?

S: $10 + {}^-16 = {}^-6$; a multiple of 3.

Ask the students to choose several other numbers for Pif and Paf and, each time, to verify that the sum is a multiple of 3.

T: So we conclude that Pif + Paf is a multiple of 3.

Relabel the string picture so that "Pif" and "Paf" are both in the 2-string.

T: Choose any numbers you wish for Pif and Paf. In which string is Pif + Paf?

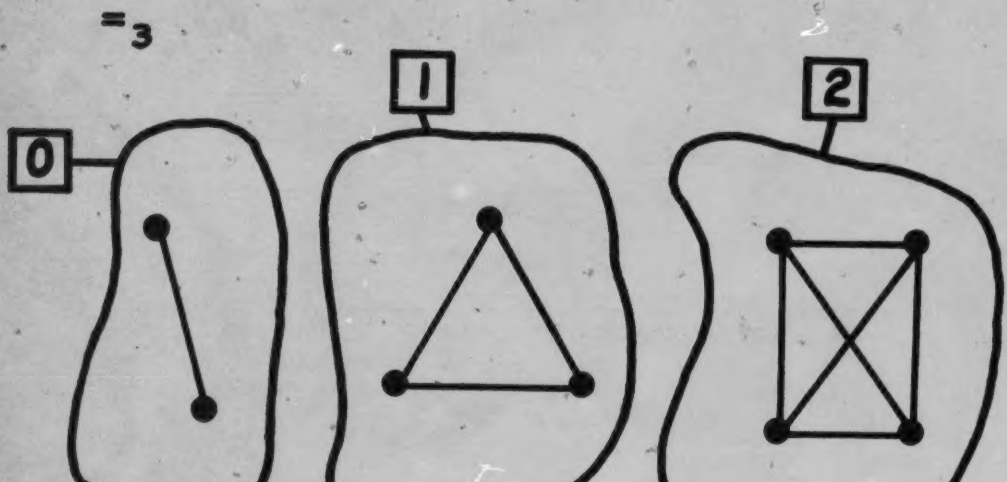
Give the students a minute or two to study this problem.

S: A number which is 2 more than a multiple of 3 plus a number which is 2 more than a multiple of 3 is a number which is 4 more than a multiple of 3, and this is just the same as a multiple of 3 plus 1.

Accept any reasonable explanation. Ask the students to choose other numbers for Pif and Paf and, each time, to verify that the sum is a multiple of 3 plus 1.

Exercise 3

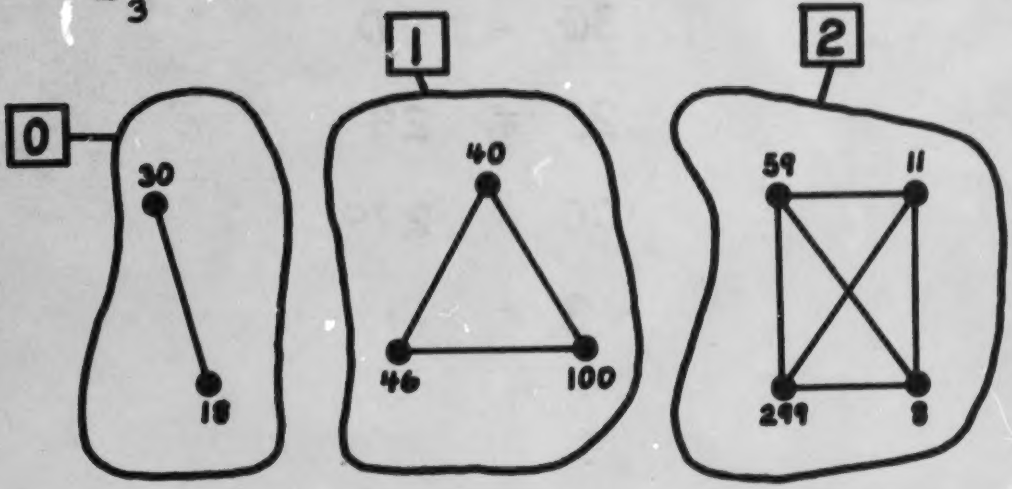
Alter your string picture so that it contains the information in the next illustration.



T: The red cords are for $=_3$ (read "equals modulo 3"). Two numbers are equal modulo 3 if and only if they are in the same string. Who can label some of the dots?

Invite students to label the dots. After a few minutes your picture might look like the one below.

$=_3$



T: Do you notice anything interesting about the numbers in our picture?

S: 46 - 40 = 6, and 6 is a multiple of 3.

T: Yes. What about the other numbers in our picture?

The students should observe that the difference of numbers inside a given string is always a multiple of 3.

T: Another way to think of the relation $=_3$ is: two numbers are equal modulo 3 if and only if their difference is a multiple of 3.

Can you give me two numbers that are not equal modulo 3?

S: 30 and 40.

N 34

Write the students' suggestions on the board.

$$30 \neq_3 40$$

$$25 \neq_3 29$$

$$100 \neq_3 299$$

$$3 \neq_3 11$$

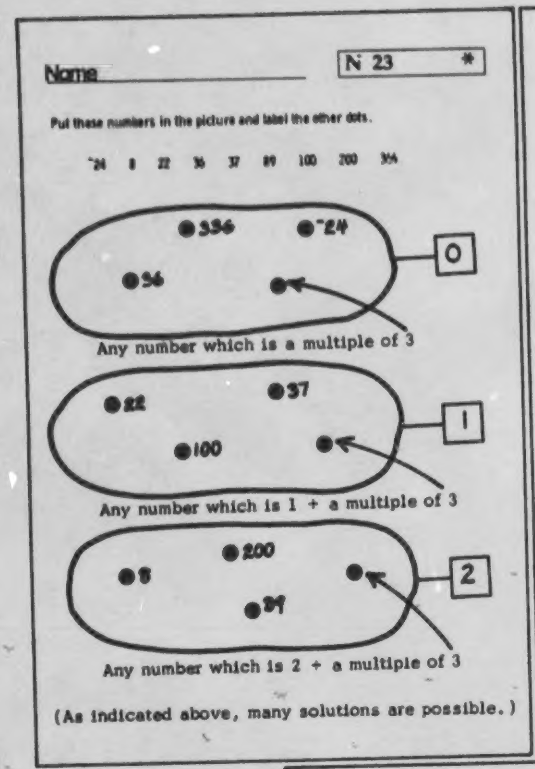
T: What do you notice about these numbers?

S: They are not in the same string. 30 is in the 0-string and 40 is in the 1-string.

S: The difference is not a multiple of 3. 11 - 3 = 8 and 8 is not a multiple of 3.

Accept any reasonable observations.

Worksheets N23, N24, and N25 are available.



Name ______________________

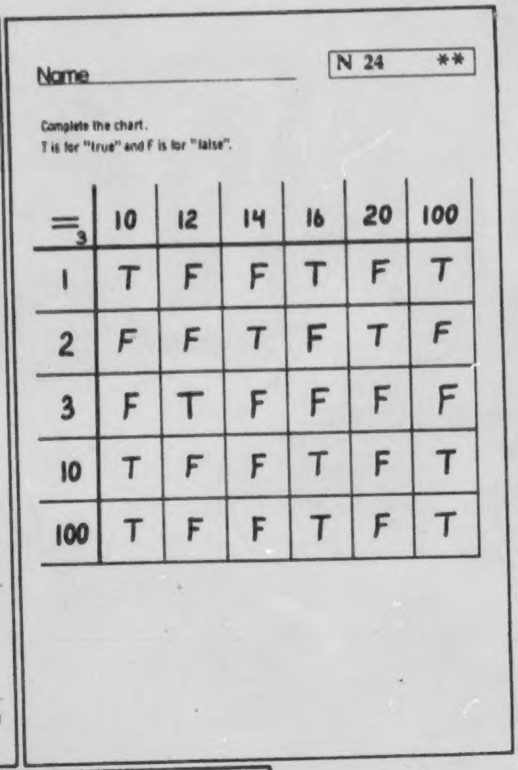
N 23 *

Put these numbers in the picture and label the other dots.

⁻24 8 22 36 37 89 100 200 336

336
⁻24
36
0
Any number which is a multiple of 3
22
37
100
1
Any number which is 1 + a multiple of 3
200
8
89
2
Any number which is 2 + a multiple of 3

(As indicated above, many solutions are possible.)

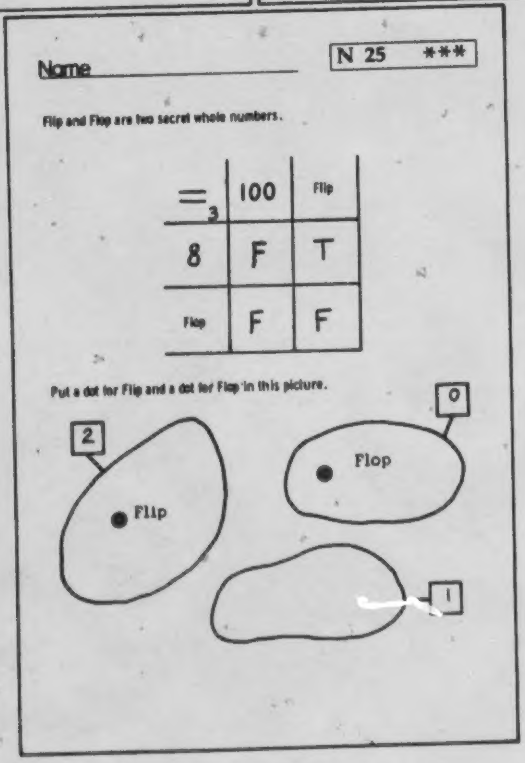


Name ______________________

N 24 **

Complete the chart.
T is for "true" and F is for "false".

$=_3$	10	12	14	16	20	100
1	T	F	F	T	F	T
2	F	F	T	F	T	F
3	F	T	F	F	F	F
10	T	F	F	T	F	T
100	T	F	F	T	F	T



Name ______________________

N 25 ***

Flip and Flop are two secret whole numbers.

$=_3$	100	Flip
8	F	T
Flop	F	F

Put a dot for Flip and a dot for Flop in this picture.

2
Flip
0
Flop
1

3

RATIONAL NUMBERS*

In this chapter, a variety of situations provide opportunities to further explore the rational numbers and to begin developing algorithms for operating with rational numbers. The situations involve arrow diagrams, games with rational numbers on the number line, hand-calculator sentences, and geometric partitioning of regions. Some of the many properties being reinforced by the activities in the lessons include:

- $\frac{4}{5}\times$ is the composite function $4\times$ followed by $\div 5$ (or "against" $5\times$) and the converse of $\frac{4}{5}\times$ is $\frac{5}{4}\times$.
- the composition of $\frac{2}{3}\times$ followed by $\frac{5}{7}\times$ is $\frac{10}{21}\times$, and $\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$.
- a rational number may have many fractional names and many decimal names.
- addition of rational numbers can often be checked by first using fractions and then using decimal names or vice versa; e.g., $\frac{2}{5} + \frac{3}{2} = \frac{4}{10} + \frac{15}{10} = \frac{19}{10} = 1.9 = 0.4 + 1.5$.

The hand-calculator supports the relationship between division and fractions and provides opportunities to do addition calculations, presented with fractions, by using the corresponding decimal names.

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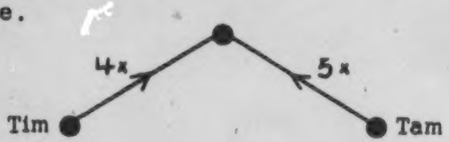
* The rational numbers include integers and all numbers which have fractional names with integer numerators and positive integer denominators; e.g., $\frac{3}{4}$, $\frac{81}{732}$, $\frac{9}{5}$, $\frac{^-1}{2}$, $\frac{^-7}{3}$. Decimal numbers are also rational numbers since, for example, $0.75 = \frac{3}{4}$, $1.3 = \frac{13}{10}$, and so on. In communicating with students, we suggest you ignore the distinction between rational numbers and their fractional names and refer to them simply as "fractions". The lesson descriptions use "fraction" to refer to a certain kind of name for a rational number, but we feel it is better pedagogical practice at this level not to force this distinction in student discussions, and we feel it is unnecessary to introduce the term "rational number" with the students.

Several situations involve comparing and ordering of rational numbers. These include putting rational numbers on the number line both using decimal names and fractional names.

CAPSULE LESSON SUMMARIES

Section 3.1: Find pairs of numbers which could be Tim and Tam in this arrow picture.



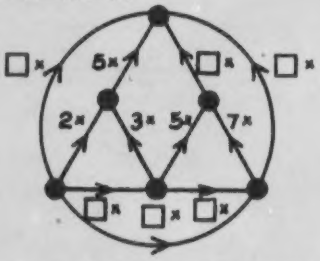
Observe that an arrow from Tim to Tam could be for $\frac{4}{5}\times$ and that its return arrow would be $\frac{5}{4}\times$. Find pairs of multiplication functions for similar arrow pictures where Tim and Tam are given but the arrow labels are not. Examine many different names for the function $\frac{2}{3}\times$.

(Worksheets N26, N27, N28, N29)

Section 3.2: Play the Newtonian Quiz game with the secret number in the thousandths. Determine which operation symbols are hidden in the blank boxes of hand-calculator sentences involving decimals.

(Worksheets N30, N31, N32, N33)

Section 3.3: Review the composition of multiplication functions to label arrows in pictures such as:



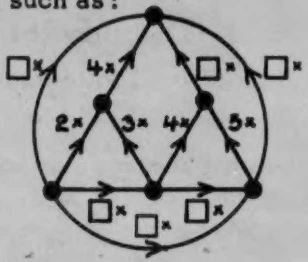
Use such situations to discuss multiplication of rational numbers.

(Worksheets N34, N35, N36, N37)

Section 3.4:	Label arrows in situations involving the addition of decimal numbers. Locate some decimals on the number line. Given two kinds of graduations on the number line, label marks on one graduation using information from the other graduation where the marks are labeled. (Worksheets N38, N39, N40)
Section 3.5:	Introduce the story of Sasquatch or Big Foot and a children's game on Big Foot's tracks. Use this situation to locate rational numbers on the number line and to order a set of rational numbers. In this situation, discuss many names for $\frac{1}{2}$.
Section 3.6:	Review the story of Sasquatch and the game (from Section 3.5) the children played using the footprints. Locate various rational numbers on the number line and observe, for example, that $\frac{2}{6} = \frac{1}{3}$ and $\frac{12}{6} = \frac{6}{3} = 2$. In this situation, compare rational numbers and determine which numbers can occur as denominators of fractions for $\frac{1}{3}$ and for $\frac{1}{4}$. For example, observe that $\frac{1}{4} < \frac{1}{3}$ but $\frac{2}{4} > \frac{1}{3}$, and $\frac{1}{3} = \frac{8}{24} = \frac{2}{6} = \frac{4}{12} = \ldots$, and $\frac{1}{4} = \frac{6}{24} = \frac{3}{12} = \frac{5}{20} = \ldots$. Determine which numbers occur both as denominators of fractions for $\frac{1}{3}$ and $\frac{1}{4}$.
Section 3.7:	Solve hand-calculator sentences both using and not using memory keys. Insert parentheses in related number sentences to indicate how the calculations are done on the calculator. Relate functions with division and find decimal names on the calculator. Then use the calculator to do calculations such as "$\frac{5}{2} + \frac{4}{10}$".

Section 3.8: Find many fractional names for rational numbers such as $\frac{1}{2}$ and $\frac{2}{3}$. Solve addition problems such as "$\frac{1}{2} + \frac{3}{4} = ?$" and "$\frac{2}{3} + \frac{3}{4} = ?$" using appropriate partitions of regions. (Worksheets N41, N42)

Section 3.9: Review composition of multiplication functions to label arrows in pictures such as:



Write fractions for certain decimals and multiply decimals by first finding a fraction for each decimal number. (Worksheets N43, N44, N45, N46)

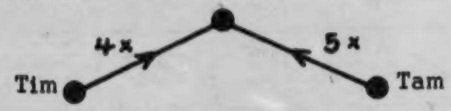
Section 3.10: Find the numbers which can be named by inserting parentheses in the expression "7 - 5 × 2 - 2". Solve hand-calculator sentences both using and not using memory keys. Insert parentheses in related number sentences to indicate how the calculations are done on the calculator. Solve problems such as "((8 × 7) - 6) - (9 × (15 - 9)) =   " on the calculator using the memory feature. Use the calculator to do calculations such as "$\frac{3}{2} + \frac{1}{5}$".

Section 3.11: Solve addition problems such as "$\frac{2}{5} + \frac{3}{4} = ?$" and "$\frac{5}{3} + \frac{3}{5} = ?$" first using appropriate partitions of regions. Then do similar calculations only thinking about the partitioning method, but finding appropriate (useful) fractional names for the numbers being added. (Worksheets N47, N48, N49)

3.1 Rational Numbers and Multiplication Functions

CAPSULE LESSON SUMMARY

Find pairs of numbers which could be Tim and Tam in this arrow picture.



Observe that an arrow from Tim to Tam could be for $\frac{4}{5}x$ and that its return arrow would be $\frac{5}{4}x$. Find pairs of multiplication functions for similar arrow pictures where Tim and Tam are given but the arrow labels are not. Examine many different names for the function $\frac{2}{3}x$ as the composite of two functions.

MATERIALS

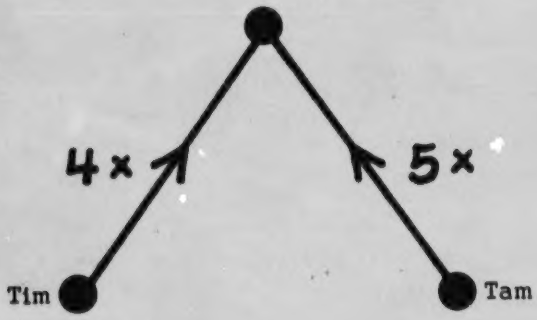
Teacher: Colored chalk

Student: Paper; pencil; Worksheets N26, N27, N28, N29

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.



T (tracing the 4x arrow and then the 5x arrow): The arrow picture shows that four times Tim equals five times Tam. Find some pairs of numbers that Tim and Tam could be.

As students suggest them, make a list on the board of pairs of numbers that Tim and Tam could be. We cannot predict your students' responses, so we list here some of the first responses that were made in one of our development classes.

Tim	5	10	15	25			
Tam	4	8	12	20			

Challenge the students with questions like the following and add the appropriate pair of numbers to your chart each time.

T: If Tim is 0.5, which number is Tam? Write it on your paper.

Let the students have a moment to consider this problem.

S: Tam is 0.4, because 4 × 0.5 = 2 and 5 × 0.4 = 2.

T: If Tim is 1, which number is Tam?

S: Tam is 0.8, because 4 × 1 = 4 and 5 × 0.8 = 4.

T: If Tim is 0.1, which number is Tam?

S: Tam is 0.08, because 4 × 0.1 = 0.4 and 5 × 0.08 = 0.40.

T: Very good. Look closely at the chart. Do you notice any interesting patterns.

If suggestions are not forthcoming, highlight any two pairs of numbers from your chart using colored chalk.

Tim	5	10	15	25	0.5	1	0.1
Tam	4	8	12	20	0.4	0.8	0.08

T: I have chosen two pairs of numbers: 5 and 4, 15 and 12.
How many times 5 is 15?

S: Three times.

T: And how many times 4 is 12?

S: Three times.

Pick another two pairs of numbers from your chart. Make your choice to best suit the numerical ability of your class.

Tim	5	10	15	25	0.5	1	0.1
Tam	4	8	12	20	0.4	0.8	0.08

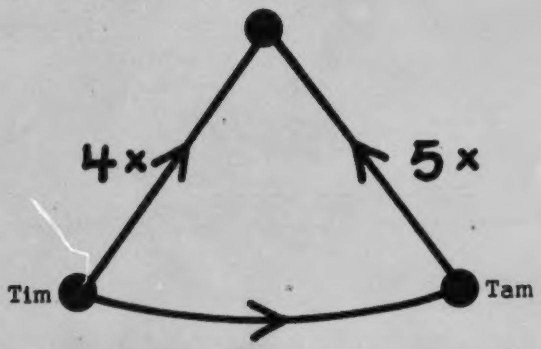
T: Now I pick another two pairs of numbers: 0.5 and 0.4, 0.1 and 0.08.
What do you notice about these numbers?

Let the students have a minute to consider the problem.

S: If 0.1 and 0.08 are both multiplied by 5, the result is 0.5 and 0.4.

If the class has not observed this multiplying idea, repeat the activity using another two pairs of numbers.

Add this blue arrow to your arrow picture.



128

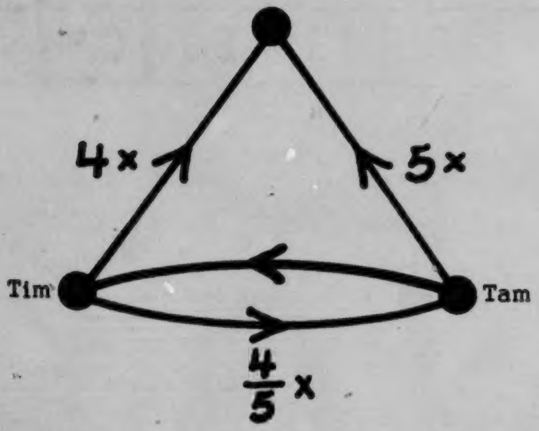
T: What could this blue arrow be for?

For emphasis, trace the 4 x arrow and "against" the 5 x arrow. Then trace the blue arrow.

S: $\frac{4}{5}$x.

T: Very good, 4 x followed by "against" 5 x is $\frac{4}{5}$x.

Draw a return arrow from Tam to Tim in another color. (Black is used in the next illustration.)



T: What could this return arrow be for?

For emphasis, trace the 5 x arrow followed by "against" the 4 x arrow. Then trace the black arrow.

S: $\frac{5}{4}$x.

Write the following problems on the board and ask the students to copy and to solve them. The correct answers are in boxes.

$\frac{4}{5} \times 15 = \boxed{12}$

$\frac{4}{5} \times 1.5 = \boxed{1.2}$

$\frac{4}{5} \times 25 = \boxed{20}$

$\frac{4}{5} \times 2.5 = \boxed{2}$

$\frac{4}{5} \times 2 = \boxed{1.6}$

$\frac{4}{5} \times 0.2 = \boxed{0.16}$

$\frac{5}{4} \times 100 = \boxed{125}$

$\frac{5}{4} \times 10 = \boxed{12.5}$

$\frac{5}{4} \times 2 = \boxed{2.5}$

$\frac{5}{4} \times 0.2 = \boxed{0.25}$

After students have had a few minutes to work on these problems independently, check the answers collectively.

T: How did you calculate "$\frac{4}{5} \times 15$"?

S: $4 \times 15 = 60$ and $60 \div 5 = 12$, so $\frac{4}{5} \times 15 = 12$.

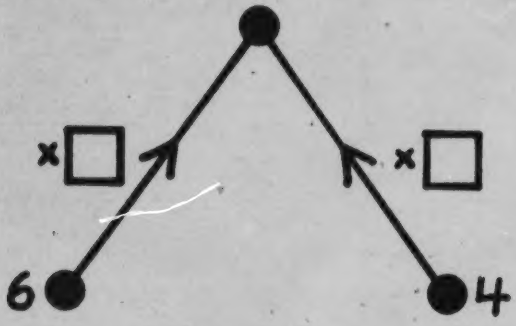
S: I divided by 5 first. $15 \div 5 = 3$ and $3 \times 4 = 12$, so $\frac{4}{5} \times 15 = 12$.

Continue in this manner for each of the remaining problems.

N 4

Exercise 2

Draw this arrow picture and chart on the board.



→								
→								

T (tracing the red arrow and then the blue arrow): What could the red arrow and the blue arrow be for? Six times the number in the red box equals four times the number in the blue box.

As students respond, let the class check the solutions. Fill in the chart as you go. If no one suggests using (non-integer) decimal numbers, do so yourself.

T: If the red arrow is for × 0.5, what could the blue arrow be for?

S: 6 × 0.5 = 3 and 4 × 0.75 = 3, so the blue arrow is for × 0.75.

Your chart might look like this upon completion of the activity.

→	×6	×4	×10	×2	×8	×20	×2,000	×0.5
→	×9	×6	×15	×3	×12	×30	×3,000	×0.75

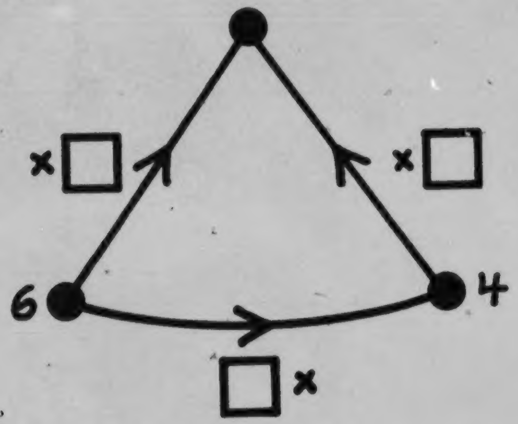
T: Do you notice anything interesting about this chart?

Let the students have a few minutes to observe the chart. Accept any reasonable explanations that are offered.

S: It is just like the last problem. Take 4 and 6 and halve each one; you get 2 and 3. 2 and 3 are another pair of numbers in the table.

S: Take 20 and 30 and multiply each by 100; you get another pair of numbers in the table -- 2,000 and 3,000.

Draw an arrow in some other color starting at 6 and ending at 4. (Black is used in the next illustration.)



T: What could this arrow be for?

If necessary, trace the red arrow and "against" the blue arrow. Then trace the black arrow.

S: $\frac{2}{3}$x.

S: $\frac{8}{12}$x.

S: $\frac{2{,}000}{3{,}000}$x.

N 4

S: $\times 0.666666\ldots = \times 0.\overset{*}{6}.$

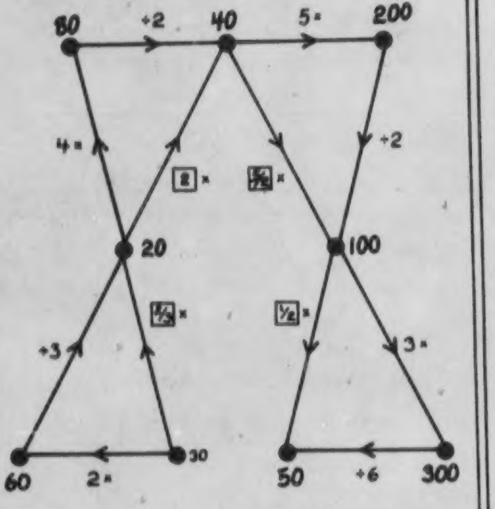
Let the students give as many names as they care to as you list them on the board in a string of equalities.

$$\times 0.\overset{*}{6} = \frac{2}{3}\times = \frac{8}{12}\times = \frac{2,000}{3,000}\times = \frac{12}{18}\times = \frac{6}{9}\times = \cdots$$

Worksheets N26, N27, N28, and N29 are available.

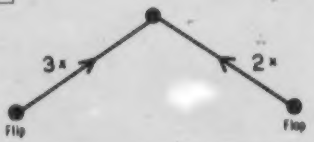
Name ____________ **N 26** *

Label each dot and fill in the box for each blue arrow.


80
÷2
40
5×
200
4×
2 ×
5/2 ×
÷2
20
100
1/3 ×
1/2 ×
÷3
3×
30
60
2×
50
÷6
300

Name ____________ **N 27** **

First clue

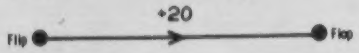

3×
2×
Flip
Flop

Fill in this chart with pairs of numbers that Flip and Flop could be.

Flip	2	4	6	10	20	12	40	80	14	16	200	600
Flop	3	6	9	15	30	18	60	120	21	24	300	900

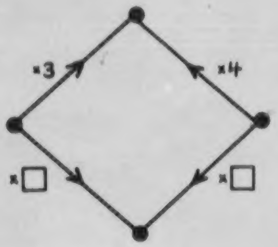
(Many other solutions are possible.)

Second clue


+20
Flip
Flop

Flip is 40. Flop is 60.

Name ____________ **N 28** ***



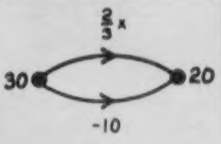
Find some ways to label the red arrow and the blue arrow. Many solutions are possible. Write some of them in this chart.

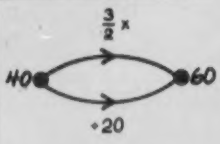
×☐ →	6	9	12	15	18	21	24	27	30	33
×☐ →	8	12	16	20	24	28	32	36	40	44

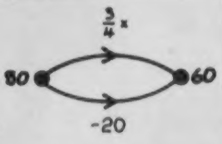
(Many other solutions are possible.)

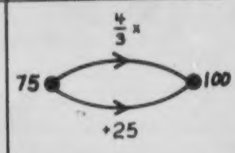
Name ____________ **N 29** ****

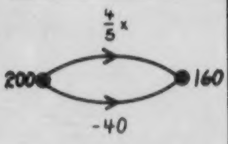
Label the dots.

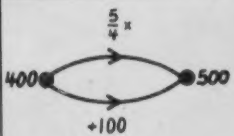









4/5 ×
200
160
-40


5/4 ×
400
500
+100

N 7

3.2 Newtonian Quiz

CAPSULE LESSON SUMMARY

Play the Newtonian Quiz game with the secret number in the thousandths. Determine which operation symbols are hidden in the blank boxes of hand-calculator sentences involving decimals.

MATERIALS

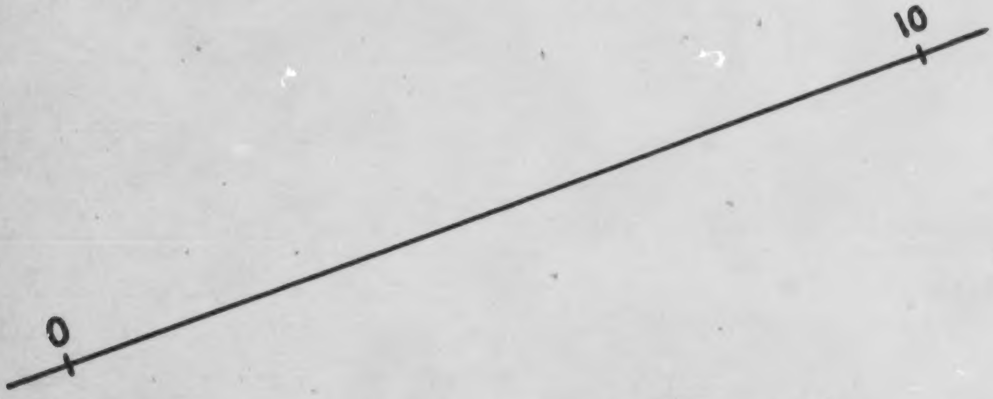
Teacher: Colored chalk
Student : Hand-calculator; Worksheets N30, N31, N32, N33

DESCRIPTION OF LESSON

Exercise 1

NOTE: In this exercise you and your class will play a number line game called "Newtonian Quiz". The purpose of playing the game is to explore the thousandths subgraduation, therefore, you do not actually choose the secret number until the end of the game.[*] As an example of how one may control the game and very quickly "home-in" on the thousandths subgraduations, we include the dialogue of a game that was played in one of our development classes. A careful study of this dialogue should reveal to you the method of using the game to explore quickly any subgraduation you wish.

Draw this part of the number line on the board.



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[*] There is a small amount of dishonesty here, since the secret number is not chosen until the end. This allows us to play a shorter and more interesting game.

T: I am thinking of a number between 0 and 10. You must try to guess this number. Who would like to make the first guess?

S: 6.

Call on a student to point to the approximate location of 6.

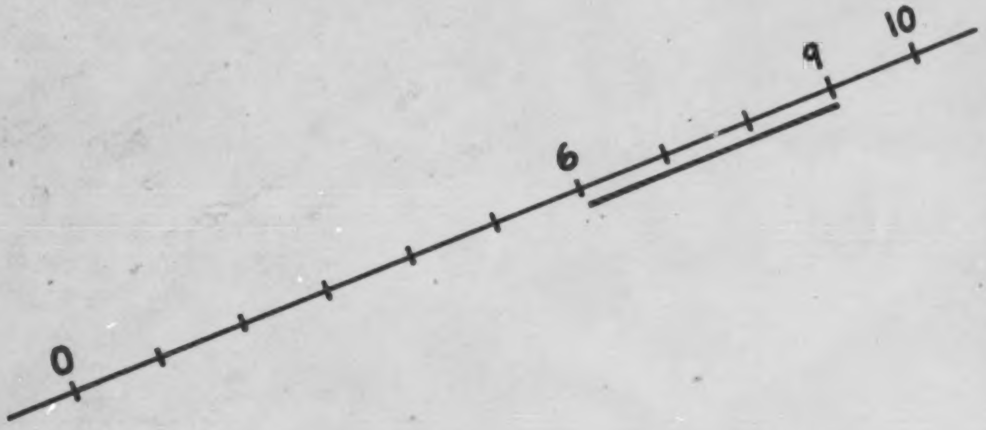
T: How could we better locate 6?

S: Divide the segment between 0 and 10 into ten smaller segments of the same length.

Perform the subdivision and invite a student to place 6 on the number line.

T: My response to your guess is 9. Now we know the secret number is between 6 and 9.

Place 9 on your number line and draw a red line segment as in the next illustration. At this point the class' attention is focused on a segment of the number line between 6 and 9. Another response you might make is 3; then you would focus attention on the segment between 3 and 6.



S: I will guess $7\frac{1}{2}$.

N 7

T: Can you give me a decimal name for $7\frac{1}{2}$?

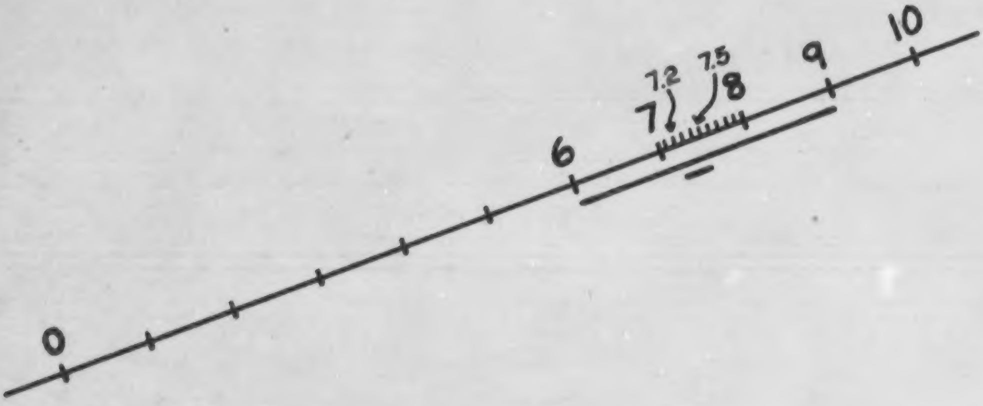
S: 7.5.

Let a student locate 7.5 on the number line.

T: My response is 7.2 (7.8 would also be a good response). How can we locate 7.2 on the number line?

S: Divide the segment between 7 and 8 into ten smaller segments of the same length.

Add the tenths graduation to your drawing and let a student locate 7.2.



S: I guess 7.4.

T: My response to your guess is 7.3.

Invite students to locate 7.4 and 7.3 on the number line.

S: My guess is 7.35.

T: Can you locate it on the number line?

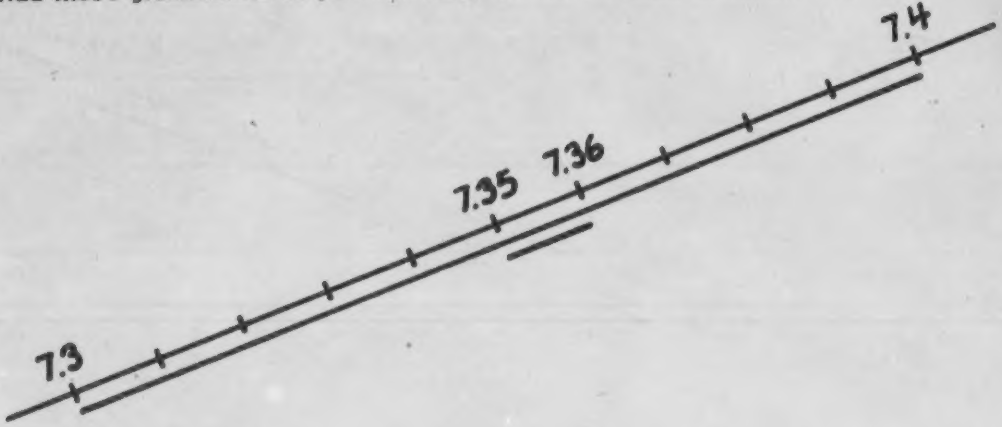
S: 7.35 is half-way between 7.3 and 7.4.

Call on a student to point to the (approximate) location for 7.35.

T: My response is 7.36 (7.34 would also be an appropriate response). Can you locate 7.36 on the number line?

S: Divide the segment between 7.3 and 7.4 into ten smaller segments of the same length.

Indicate to the class that you are magnifying the picture between 7.3 and 7.4. Add these graduations to your drawing and let a volunteer locate 7.36.



T: Now we know the secret number is between 7.35 and 7.36. What could my secret number be?

S: 7.353.

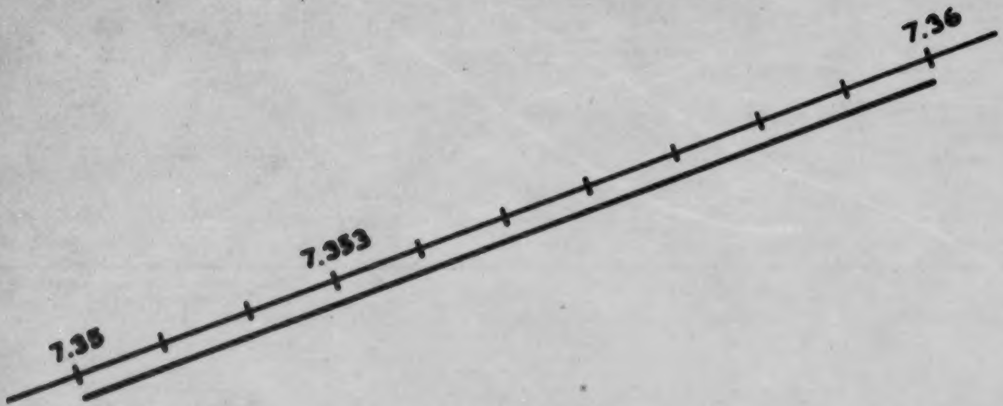
T: Can you locate 7.353 on the number line?

S: Divide the segment between 7.35 and 7.36 into ten smaller segments of the same length.

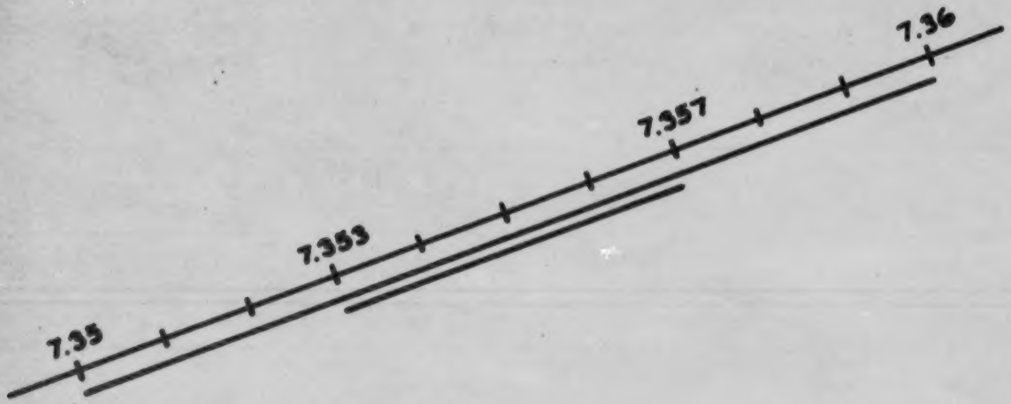
T: Good. Let's magnify the number line and locate 7.353.

N 7

Draw another segment, subgraduate it and locate 7.353.

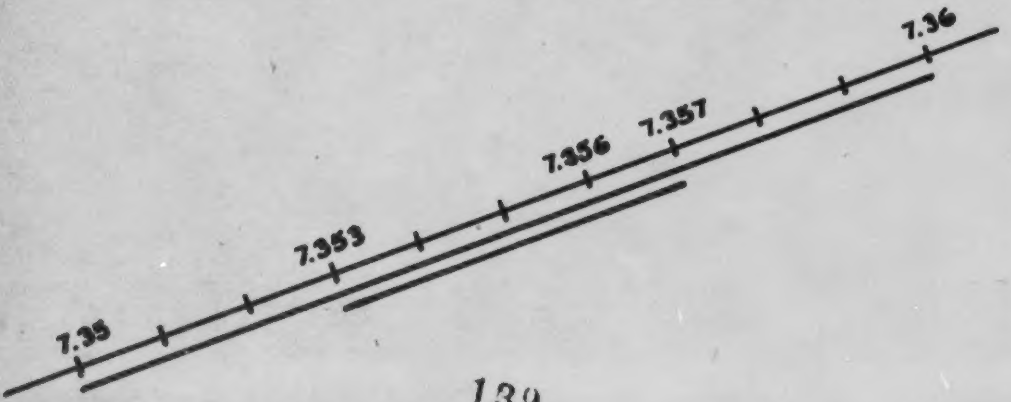

7.35
7.353
7.36

T: My response is 7.357. Can you locate 7.357?


7.35
7.353
7.357
7.36

S: My guess is 7.356 (or any guess between 7.353 and 7.357).

T: That's my secret number. Locate this number on the number line.


7.35
7.353
7.356
7.357
7.36

139

Exercise 2

Provide each student with a hand-calculator.

NOTE: We assume that the hand-calculators being discussed do chain operations from left to right and have automatic constants. (See the General Introduction, Section 2(f) for a more detailed commentary on hand-calculators.)

Write this expression on the board.

$$\boxed{3}\ \boxed{}\ \boxed{7}\ \boxed{}\ \boxed{2}\ \boxed{=}\ \boxed{10.5}$$

T: Each box hides one of the symbols: +, −, ×, or ÷. The same symbol could be in both boxes. Which symbols go in these two boxes?

Let students have a few minutes to explore this situation with their hand-calculators.

S: Put "×" in the first box and "÷" in the second box, because $3 \times 7 = 21$ and $21 \div 2 = 10.5$.

Continue this activity using some of the hand-calculator sentences given in the next illustration. Solutions are printed in red.

$$\boxed{6}\ \boxed{\div}\ \boxed{6}\ \boxed{\div}\ \boxed{4}\ \boxed{=}\ \boxed{0.25}$$

$$\boxed{10}\ \boxed{\times}\ \boxed{0.5}\ \boxed{-}\ \boxed{1.5}\ \boxed{=}\ \boxed{3.5}$$

$$\boxed{27}\ \boxed{\div}\ \boxed{9}\ \boxed{\div}\ \boxed{6}\ \boxed{=}\ \boxed{0.5}$$

N 7

$$\boxed{1.2}\ \boxed{+}\ \boxed{2.8}\ \boxed{\times}\ \boxed{5}\ \boxed{=}\ \boxed{20}$$

$$\boxed{3.2}\ \boxed{-}\ \boxed{0.7}\ \boxed{\div}\ \boxed{2}\ \boxed{=}\ \boxed{1.25}$$

$$\boxed{3}\ \boxed{\times}\ \boxed{4}\ \boxed{\div}\ \boxed{5}\ \boxed{=}\ \boxed{2.4}$$

$$\boxed{3}\ \boxed{\div}\ \boxed{4}\ \boxed{\times}\ \boxed{5}\ \boxed{=}\ \boxed{3.75}$$

Write this expression on the board.

$$\boxed{12}\ \boxed{}\ \boxed{4}\ \boxed{+}\ \boxed{=}\ \boxed{=}\cdots\boxed{1{,}000}$$

T: For this hand-calculator sentence you must press $\boxed{12}$; then one of the symbol keys : $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$; then $\boxed{4}$; then $\boxed{+}$; and then $\boxed{=}$ as many times as necessary until 1,000 appears on the display. Which symbol could be hidden in the box?

Let the students have a few minutes to explore this situation.

S: "-" goes in the blank box. If I press $\boxed{12}\boxed{-}\boxed{4}\boxed{+}\boxed{=}\boxed{=}$ and so on, 1,000 will appear on the display.

T: Very good. How many times must $\boxed{=}$ be pressed?

S: 125 times, because 125 × 8 = 1,000.

NOTE: On some hand-calculators $\boxed{=}$ would only need to be pressed 124 times, because the first time $\boxed{=}$ is pressed 8 + 8 or 2 × 8 is on the display. For a more detailed discussion of this situation see the General Introduction, Section 2(f).

Repeat the activity using this hand-calculator sentence.

$$\boxed{0.2}\ \boxed{}\ \boxed{5}\ \boxed{+}\ \boxed{=}\ \boxed{=}\cdots\boxed{1{,}000}$$

S: "x" could be in the blank box, because 1,000 will appear on the display if I press $\boxed{0.2}$ $\boxed{\times}$ $\boxed{5}$ $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ and so on.

T: How many times must $\boxed{=}$ be pressed?

S: 1,000 times, because 1,000 × 1 = 1,000.

There is another solution using the $\boxed{\div}$ key and pressing $\boxed{=}$ 25,000 times, because 25,000 × 0.04 = 1,000. If this solution is suggested accept it, however it is not necessary to mention it at this time.

Worksheets N30, N31, N32, and N33 are available.

Name ____________ **N 30 ***

Put one of the symbols: + , × , − , ÷ in each blank box to make the hand-calculator sentence true. A symbol may be used twice in one sentence.

4 × 5 − 12 = 8

18 ÷ 3 × 5 = 30

22 + 18 ÷ 4 = 10

107 − 7 ÷ 100 = 1

7 ÷ 7 + 7 = 8

6 + 6 − 6 = 6

(There are several other solutions.)

6 × 7 − 7 = 35

Name ____________ **N 31 ****

Put one of the symbols: + , × , − , ÷ in each blank box to make the hand-calculator sentence true. A symbol may be used twice in one sentence.

6 × 0.5 + 5 = 8

10 × 2.4 ÷ 8 = 3

3 ÷ 6 × 100 = 50

3.7 + 2.8 × 2 = 13

6.2 − 1.7 × 4 = 18

1 ÷ 2 + = = … 236.5

0.25 × 8 + = = … 2,978

(Another solution may be obtained by replacing "x" by "÷".)

0.37 + 0.63 + = = … 8,981

Name ____________ **N 32 *****

Put one of the symbols: + , × , − , ÷ in each blank box to make the hand-calculator sentence true.

4 ÷ 8 + = = … 875

How many times do you press =? 1,750 (or 1,749)*

7.2 − 6.8 + = = … 1,000

How many times do you press =? 2,500 (or 2,499)*

10 × 0.025 + = = … 1,000

How many times do you press =? 4,000 (or 3,999)*

0.2 × 10 + = = … 1,000

(Another solution may be obtained by replacing "x" by "÷".)

How many times do you press =? 500 (or 499)*

*The number of times you press = depends on the calculator. See the General Introduction Section 2(f) for a more detailed commentary.

Name ____________ **N 33 ******

Put one of the symbols: + , × , − , ÷ in each blank box to make the hand-calculator sentence true. A symbol may be used more than once in a sentence.

4 ÷ 5 × 10 × 7 = 56

1 − 0.1 − 0.01 × 100 = 89

5 ÷ 5 ÷ 5 ÷ 5 = 0.04

2 ÷ 2 ÷ 2 + = = … 75.5

How many times do you press =? 150 (or 149)*

5 ÷ 8 × 100 + = = … 2,000

(Another solution may be obtained by interchanging "÷" and "x".)

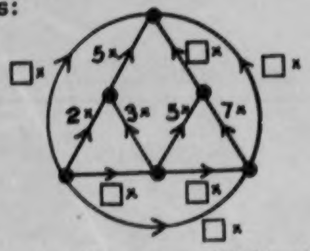
How many times do you press =? 32 (or 31)*

*The number of times you press = depends on the calculator. See the General Introduction Section 2(f) for a more detailed commentary.

3.3 Multiplication of Rational Numbers #1

CAPSULE LESSON SUMMARY

Review the composition of multiplication functions to label arrows in pictures such as:



Use such situations to discuss multiplication of rational numbers.

MATERIALS

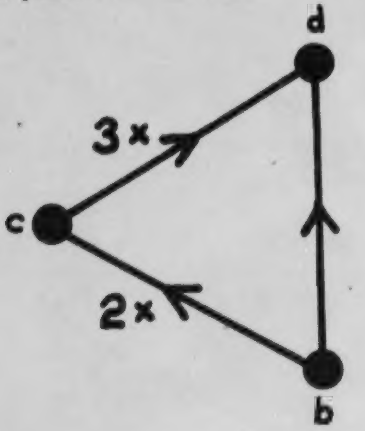
Teacher: Colored chalk

Student : Paper; colored pencils; Worksheets N34, N35, N36, N37

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: What could the blue arrow be for?

S: 6x.

N8

Label the blue arrow "6x".

T: Let's choose some numbers and see if we are correct.

Label c "14" and ask the students for the other two numbers (for b and d).

T (pointing to b): Which number is here?

S: 7, because 2 × 7 = 14.

T (pointing to d): And which number is here?

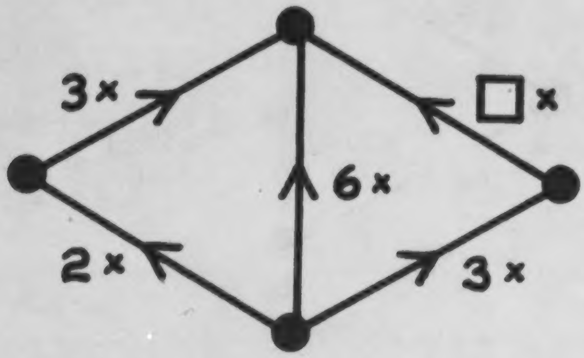
S: 42, because 3 × 14 = 42.

T (tracing the blue arrow): Is 6 × 7 = 42?

S: Yes.

Choose a few more numerical examples appropriate to the abilities of your students illustrating that the blue arrow could be 6x.

Add the two arrows shown below to your picture and ask the students to complete the picture.



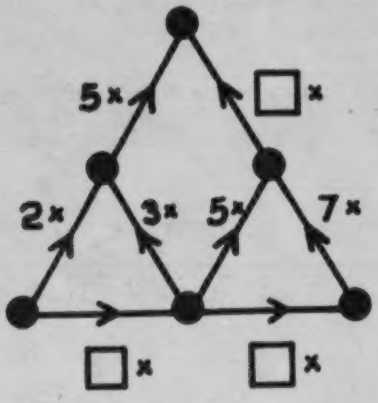
T: How could we label this red arrow? Why?

S: 2 ×, because 3 × followed by 2 × is the same as 2 × followed by 3 ×.

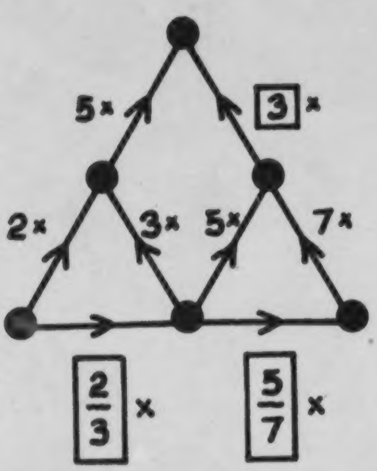
Accept any explanation along this line.

Exercise 2

Draw this arrow picture on the board.



Invite students to fill in the boxes for the arrows. Answers are provided below.



N8

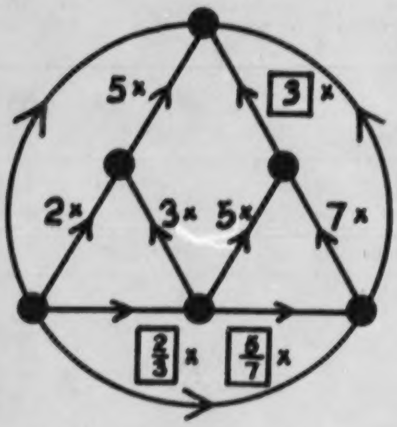
For the benefit of students who have difficulty labeling the $\frac{2}{3}\times$ arrow, lead the class to observe a composition of arrows as described below.

T (tracing "with" or "against" arrows as they are mentioned): This red arrow is 2 × followed by "against" 3 ×. What could this red arrow be for?

S: $\frac{2}{3}\times$.

Use a similar composition of arrows to aid in labeling the $\frac{5}{7}\times$ arrow. For the 3 × arrow, observe that 3 × followed by 5 × is the same as 5 × followed by 3 ×.

Add the three blue arrows shown below to your picture and ask students to label them in the order suggested in the following dialogue.



T: Look at the blue arrow to the left. It is for 2 × followed by 5 ×. What could this blue arrow be for? Why?

S: 10 ×, because 2 × 5 = 10.

T: Yes. Now consider the blue arrow to the right. It is for 7 × followed by 3 ×. What could this blue arrow be for? Why?

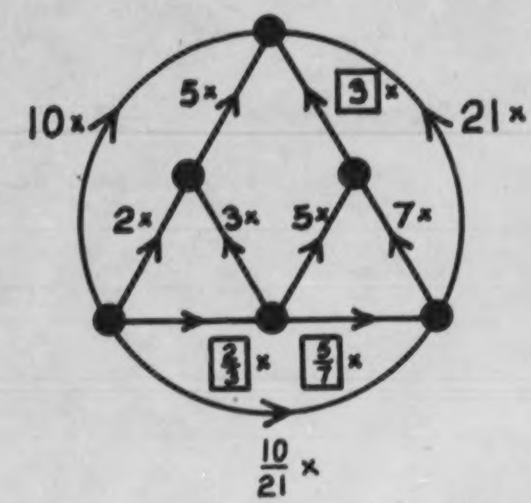
S: 21 ×, because 7 × 3 = 21.

T: Now look at the last blue arrow along the bottom of the picture. What could this blue arrow be for? Use the other two blue arrows to help you.

Let the class have a few minutes to consider the problem.

S: $\frac{10}{21}\times$.

T (tracing "with" or "against" the appropriate arrows in turn): 10 x followed by "against" 21 x is $\frac{10}{21}\times$.


5x
$\boxed{3}\times$
10x
21x
2x
3x
5x
7x
$\boxed{\frac{2}{3}}\times$
$\boxed{\frac{5}{7}}\times$
$\frac{10}{21}\times$

T (tracing the appropriate arrows as they are mentioned): We already know that 2 x followed by 5 x is 10 x and $2 \times 5 = 10$. We also know that 7 x followed by 3 x is 21 x and $7 \times 3 = 21$. Now, we see that $\frac{2}{3}\times$ followed by $\frac{5}{7}\times$ is $\frac{10}{21}\times$ or $\frac{2 \times 5}{3 \times 7}\times$.

Write this equality on the board.

$$\frac{10}{21}\times = \frac{2\times 5}{3\times 7}\times$$

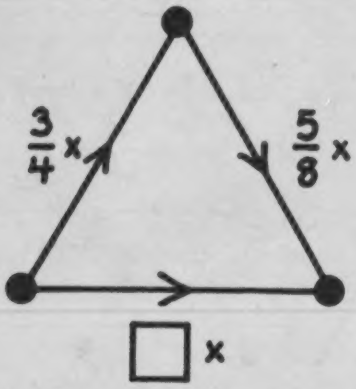
T: So, we will conclude that $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$.

N 8

Write this equality on the board.

$$\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$$

Draw this arrow picture on the board.



T: What could the blue arrow be for?

S: $\frac{15}{32}$x.

T: How did you find it?

S: $3 \times 5 = 15$ and $4 \times 8 = 32$.

T: So, $\frac{3}{4}$x followed by $\frac{5}{8}$x is $\frac{15}{32}$x.

Write this calculation on the board.

$$\frac{3}{4} \times \frac{5}{8} = \square$$

T: How could we perform this calculation?

S: $3 \times 5 = 15$ and $4 \times 8 = 32$, so $\frac{3}{4} \times \frac{5}{8} = \frac{15}{32}$.

Write this information on the board.

$$\frac{3}{4} \times \frac{5}{8} = \frac{3 \times 5}{4 \times 8} = \frac{15}{32}$$

T: To multiply two fractions; just multiply the top numbers together, then multiply the bottom numbers together.

Ask the students to copy and solve the problems below.

$$\frac{3}{5} \times \frac{2}{3} = \square$$

$$\frac{5}{8} \times \frac{4}{5} = \square$$

$$\frac{2}{5} \times \frac{3}{4} = \square$$

When it is obvious that most students have completed the problem, invite students to explain the calculations.

S: $\frac{3}{5} \times \frac{2}{3} = \frac{3 \times 2}{5 \times 3} = \frac{6}{15}$.

T: The answer to the first problem is $\frac{6}{15}$. Do you know some other names for "$\frac{6}{15}$"?

S: "$\frac{2}{5}$" or "$\frac{4}{10}$".

S: "0.4".

S: "$\frac{12}{30}$".

N 8

Continue in a similar manner checking the two remaining problems. Answers are given below.

$$\frac{5}{8} \times \frac{4}{5} = \boxed{\frac{5 \times 4}{8 \times 5} = \frac{20}{40} = \frac{2}{4} = \frac{1}{2} = 0.5}$$

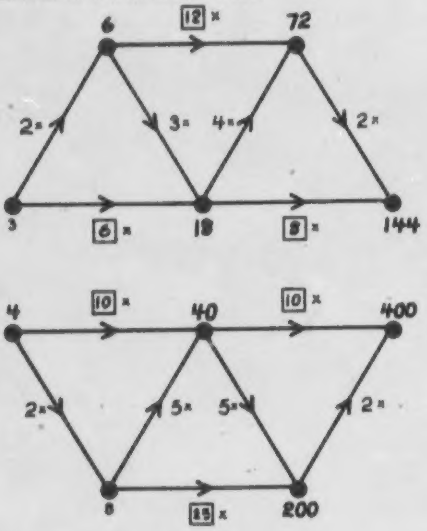
$$\frac{2}{5} \times \frac{3}{4} = \boxed{\frac{2 \times 3}{5 \times 4} = \frac{6}{20} = \frac{3}{10} = 0.3}$$

Worksheets N34, N35, N36, and N37 are available.

Name ______________________

N 34 *

Label the dots and fill in the boxes for the arrows.

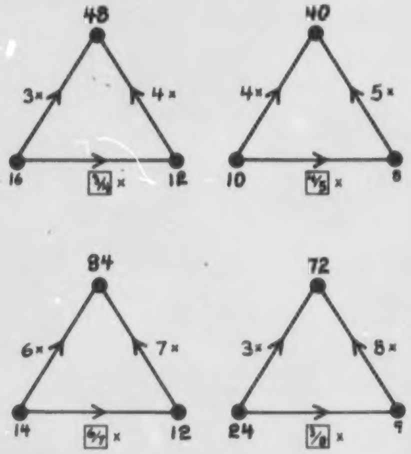


6
12 x
72
2 x
3 x
4 x
2 x
3
6 x
18
8 x
144
4
10 x
40
10 x
400
2 x
5 x
5 x
2 x
8
25 x
200

Name ______________________

N 35 **

Label the dots and fill in the boxes for the blue arrows.

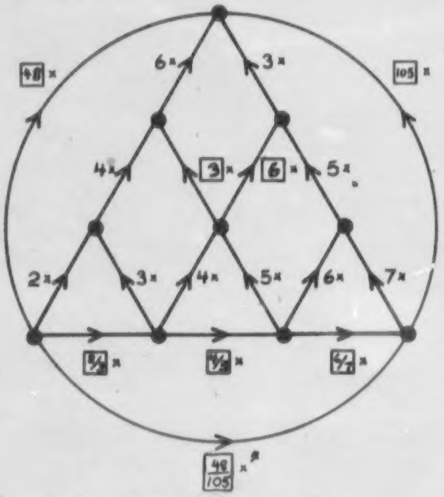


48
3 x
4 x
16
3/4 x
12
40
4 x
5 x
10
4/5 x
8
84
6 x
7 x
14
6/7 x
12
72
3 x
8 x
24
3/8 x
9

Name ______________________

N 36 ***

Fill in the boxes for the arrows.



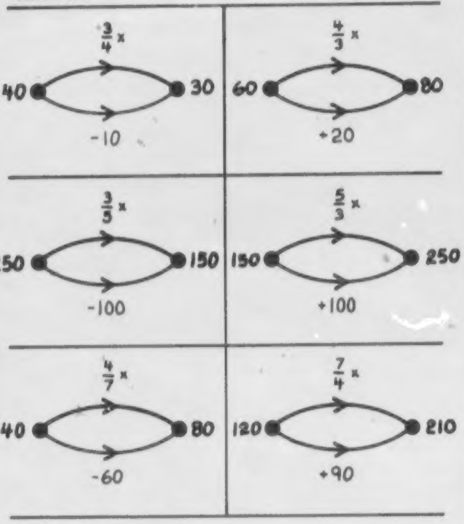
48 x
6 x
3 x
105 x
4 x
3 x
6 x
5 x
2 x
3 x
4 x
5 x
6 x
7 x
2/3 x
4/5 x
6/7 x
48/105 x #

This fractional name is the one that students most likely will give. It is possible that other correct names could be given; e.g., $\frac{48}{105} = \frac{16}{35} = \frac{32}{70} = \frac{64}{140} = \ldots$:

Name ______________________

N 37 ****

Label the dots.



3/4 x
40
30
-10
4/3 x
60
80
+20
3/5 x
250
150
-100
5/3 x
150
250
+100
4/7 x
140
80
-60
7/4 x
120
210
+90

N 15

3.4 Decimals on the Number Line

CAPSULE LESSON SUMMARY

Label arrows in situations involving the addition of decimal numbers. Locate some decimals on the number line. Given two kinds of graduations on the number line, label marks on one graduation using information from the other graduation where the marks are labeled.

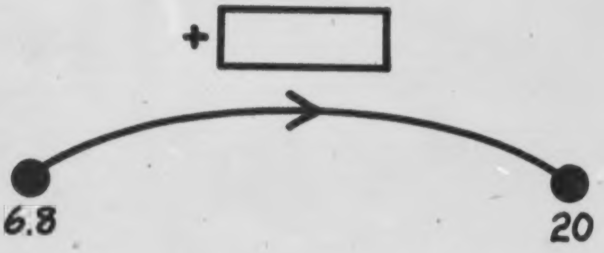
MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils; Worksheets N38, N39, N40

DESCRIPTION OF LESSON

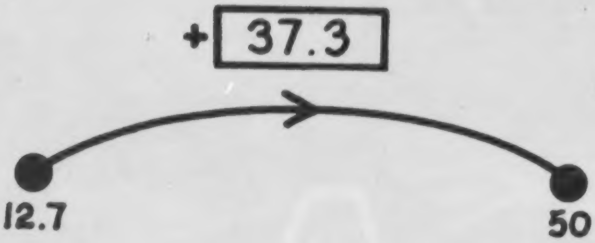
Exercise 1

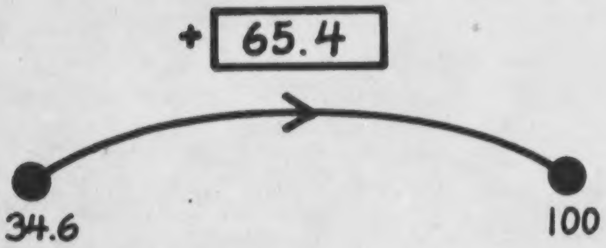
Draw this arrow picture on the board. Ask your students to copy the picture and label the arrow.

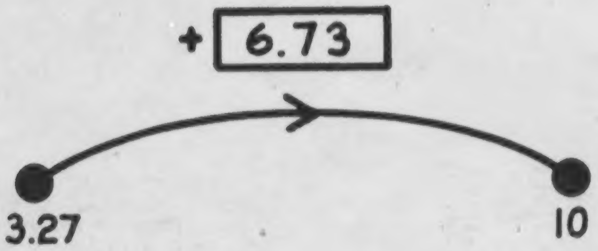


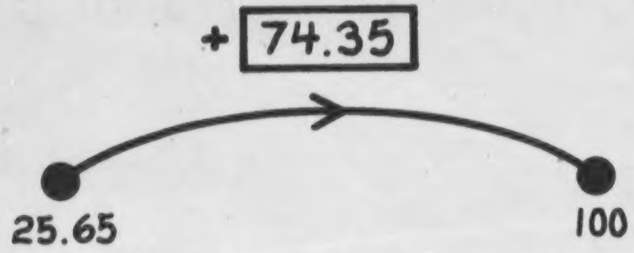
S: +13.2.

Repeat this activity several more times using problems appropriate to the numerical ability of your class. Some examples, with correct answers in the boxes, are given below. You may wish to provide hand-calculators with this exercise for students who are still weak at calculating with decimals.









T: Look at the last two problems. Do you notice an easy way of finding the answer?

While the students are considering the situation, write this information on the board.

$$\begin{array}{r} 3.27 \\ \square.\square\square \\ \hline 10.00 \end{array} \qquad \begin{array}{r} 25.65 \\ +\ \square\square.\square\square \\ \hline 100.00 \end{array}$$

N 15

S: Start from the right; 3 goes in the first box because 7 + 3 = 10. For the other two digits add a number to get 9, so 7 is in the next box and then 6.

S: The same method works for the last problem. Starting at the right, 5 is in the first box because 5 + 5 = 10. Then for the other digits, add some number to get 9 (i.e., 3, 4, and 7).

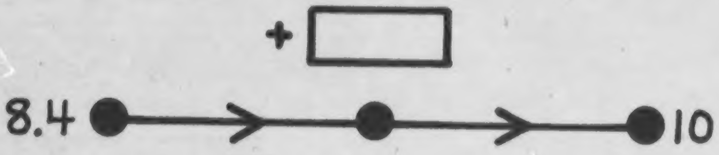
Complete the calculations on the board as students explain.

$$\begin{array}{r} 3.2\;7 \\ \boxed{6}.\boxed{7}\,\boxed{3} \\ \hline 10.0\;0 \end{array} \qquad\qquad \begin{array}{r} 2\;5.6\;5 \\ +\,\boxed{7}\,\boxed{4}.\boxed{3}\,\boxed{5} \\ \hline 1\;0\;0.0\;0 \end{array}$$

Exercise 2

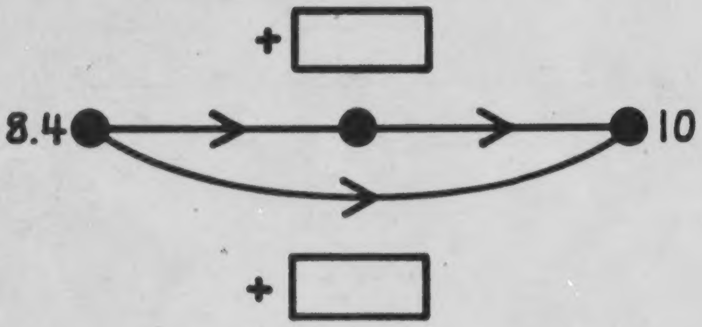
With a weak class, you may prefer to omit this exercise and continue the lesson with Exercise 3.

Draw this arrow picture on the board.



T: Copy this picture and fill in the box.

For students who need help getting started, add this blue arrow to your picture.



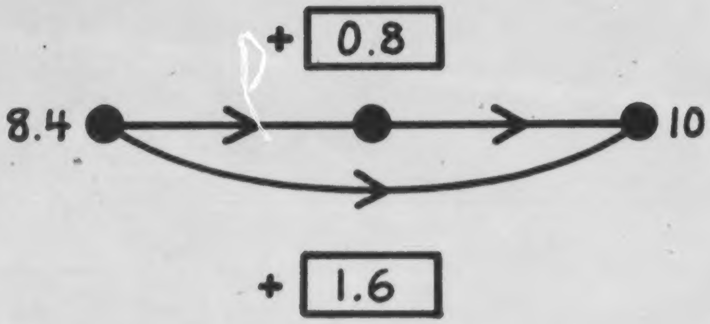
T: What could this blue arrow be for?

S: +1.6.

Label the blue arrow "+1.6".

T: Now, what could the red arrows be for?

S: +0.8.



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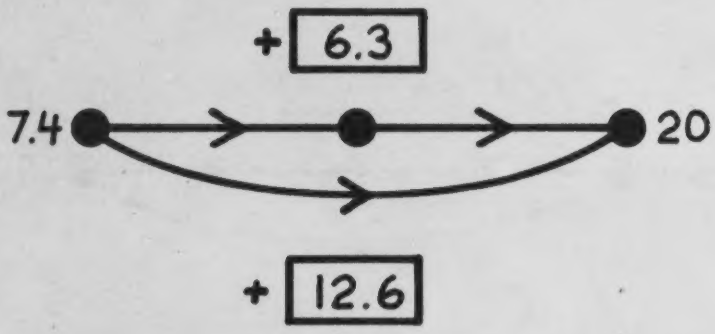
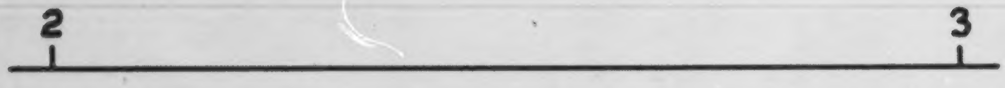
N 15

Repeat the activity using this arrow picture.



Exercise 3

Draw this part of the number line on the board.

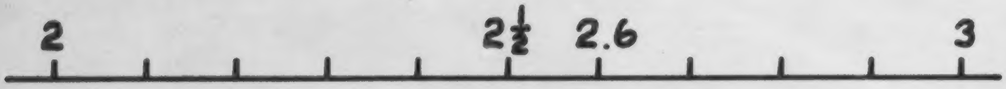


T: Who can locate $2\frac{1}{2}$ on the number line?

Invite a student to point to its location. (The midpoint of the segment.)

T: Who can locate 2.6 on the number line?

Some students will most likely suggest that the segment between 2 and 3 be divided into ten segments of the same length. Perform the sub-division and ask a student to locate 2.6.



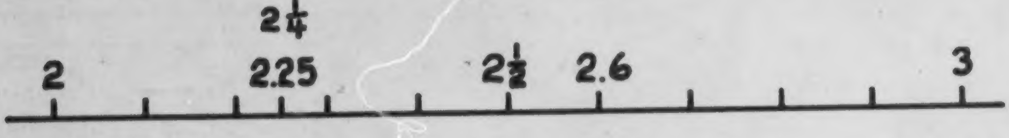
T: Who can locate $2\frac{1}{4}$ on the number line?

As a hint for this situation, ask:

T: What is a decimal name for $2\frac{1}{4}$?

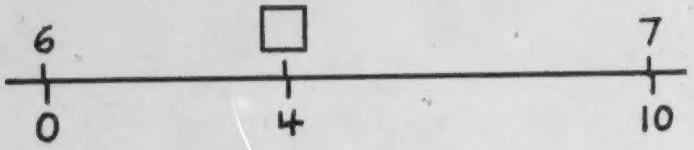
S: "2.25".

S: $2\frac{1}{4}$ is half-way between 2.2 and 2.3.



Exercise 4

Draw this picture on the board.



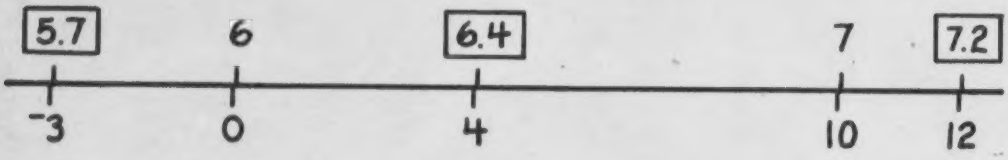
T: I have drawn a scale with two kinds of graduations. You must fill in the box on the blue graduation.

Let the students have a minute or two to consider the situation.

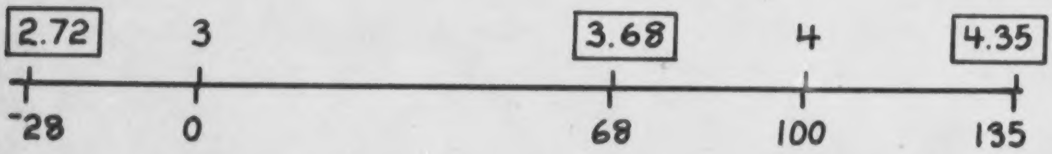
N 15

S: The segment between 6 and 7 could be divided into ten segments of equal length. The red "4" would be for the fourth segment, so the number in the box is 6.4.

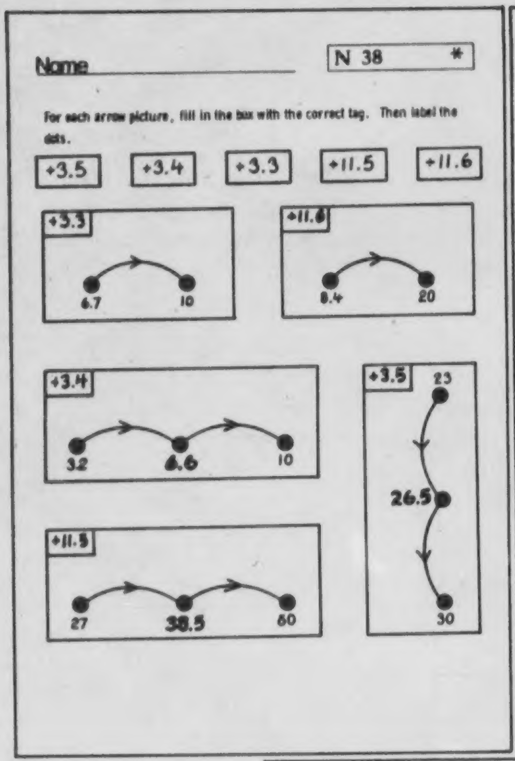
Extend your picture as shown below and ask the students to fill in the two boxes. The correct answers are provided in this picture.


5.7
6
6.4
7
7.2
⁻3
0
4
10
12

Repeat this activity using the picture below. The answers are in the boxes.


2.72
3
3.68
4
4.35
⁻28
0
68
100
135

Worksheets N38, N39, and N40 are available.

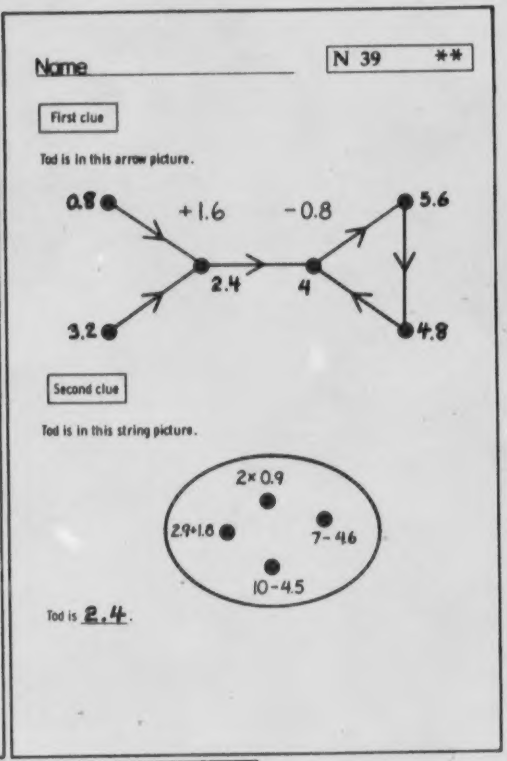


Name

N 38 *

For each arrow picture, fill in the box with the correct tag. Then label the dots.

+3.5
+3.4
+3.3
+11.5
+11.6
+3.3
6.7
10
+11.6
8.4
20
+3.4
3.2
6.6
10
+3.5
23
26.5
30
+11.5
27
38.5
50



Name

N 39 **

First clue

Tod is in this arrow picture.

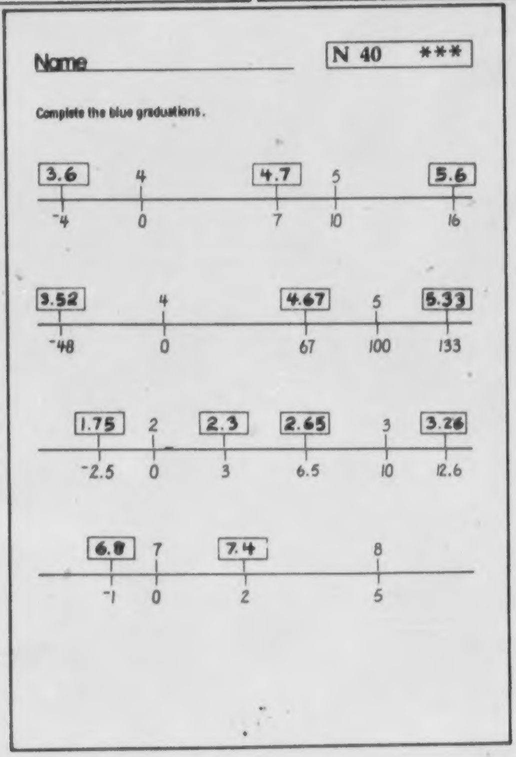
0.8
+1.6
-0.8
5.6
2.4
4
3.2
4.8

Second clue

Tod is in this string picture.

2× 0.9
2.9+1.8
7- 4.6
10-4.5

Tod is 2.4.



Name

N 40 ***

Complete the blue graduations.

3.6
4
4.7
5
5.6
⁻4
0
7
10
16
3.52
4
4.67
5
5.33
⁻48
0
67
100
133
1.75
2
2.3
2.65
3
3.26
⁻2.5
0
3
6.5
10
12.6
6.8
7
7.4
8
⁻1
0
2
5

N 17

3.5 Sasquatch #1

CAPSULE LESSON SUMMARY

Introduce the story of Sasquatch or Big Foot and a children's game on Big Foot's tracks. Use this situation to locate rational numbers on the number line and to order a set of rational numbers. In this situation, discuss many names for $\frac{1}{2}$.

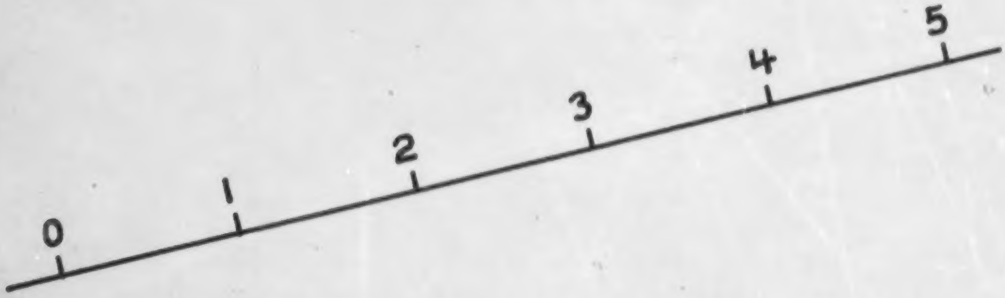
MATERIALS

Teacher: Colored chalk; chalkboard compass (optional)
Student : Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Write "SASQUATCH" at the top of the board and draw this part of the number line. (The numerals should be spaced 30 to 40 cm apart.)



T: Do you know who Sasquatch is?

S: "Sasquatch" is another name for Big Foot.

Let the students discuss the legend of Sasquatch for a few minutes.

T: According to the legend, Sasquatch or Big Foot is a primitive creature living in the northwestern part of the United States. It has not yet been proven that Big Foot actually exists, but many people believe that it does.

T: One morning some children discovered a line of foot prints in the snow. The foot prints were very large and very far apart. The children were sure the tracks were Big Foot's.

Pretend that the marks on the number line I have drawn represent the Big Foot tracks.

The children invented a game to play using the tracks. They start walking from one of the foot prints (0 in our picture). Each child is careful to take steps of the same length. Each time a child steps on one of the Big Foot tracks (at 1, 2, 3, and so on in our picture) that child scores a point.

Provide a selection of objects of different lengths; for example, an eraser, pencils, pieces of chalk, and a paper clip. Invite students to choose one of the objects to simulate a step size and then to "play" the game with the object or length of their choice, each time starting at 0 and trying to land on 1. A convenient alternative to providing a selection of objects would be to provide a chalkboard compass. The advantage of this tool is that any reasonable step-size can be chosen by opening the compass suitably.

Exercise 2

Draw this table on the board.

Child	A	B	C	D	E	F	G	H	J
Number of steps to reach 1	4	12	15	2	5	10	9	3	99

T: In this chart, there is information about some children who play the game. Which child takes the largest steps?

N 17

Let the students have a minute to study the chart.

S: D takes the largest steps. D only needs two steps to go from 0 to 1.

T: Which child takes the smallest steps?

S: J; J needs 99 steps to reach 1.

T: Who takes larger steps, H or E?

S: H; H needs three steps to reach 1 while E needs five steps. H's steps must be larger because H takes fewer steps.

T: Good. Who takes smaller steps, A or D?

S: A; A needs four steps to reach 1 and D only needs two steps. The more steps needed, the smaller the step.

T: How does A's step-size compare to D's step-size?

S: A takes two steps for each one that D takes; A's steps are half the size of D's.

Ask the students to compare several other step-sizes.

T: Consider child D. This child takes two steps to go from 0 to 1. Point to the place where D's first step lands.

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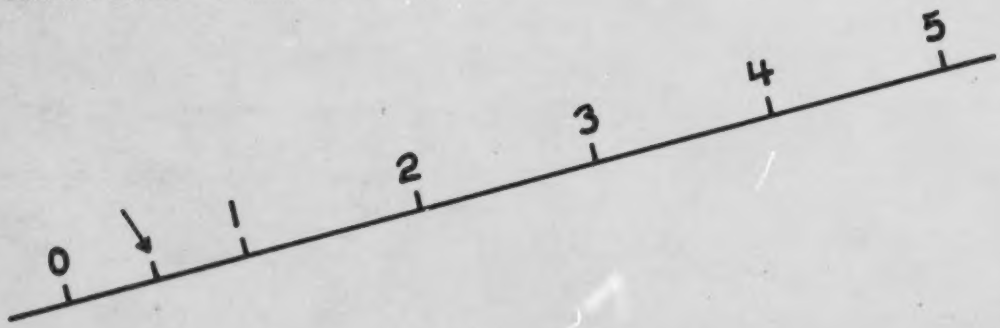
Invite a student to locate D's first step.



S: D's first step is in the middle between 0 and 1.

T: How could we label this point?

S: "$\frac{1}{2}$".

Label the point for D's first step "$\frac{1}{2}$".

T: The "1" tells us that this is the first step and the "2" tells us that two steps are needed to go from 0 to 1.

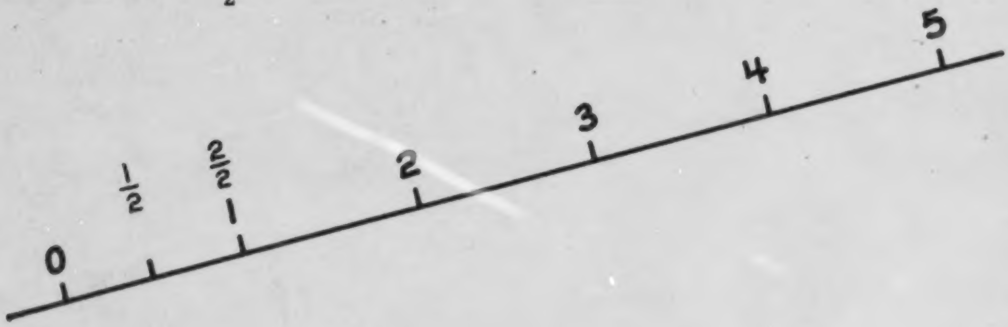
Point to the place where D's second step lands.

S: At 1.

T: How could we label this point to show D's second step lands here?

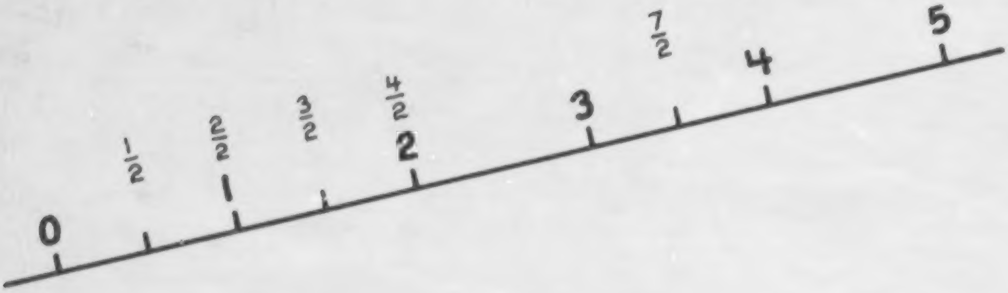
S: "$\frac{2}{2}$".

Label the point "$\frac{2}{2}$".



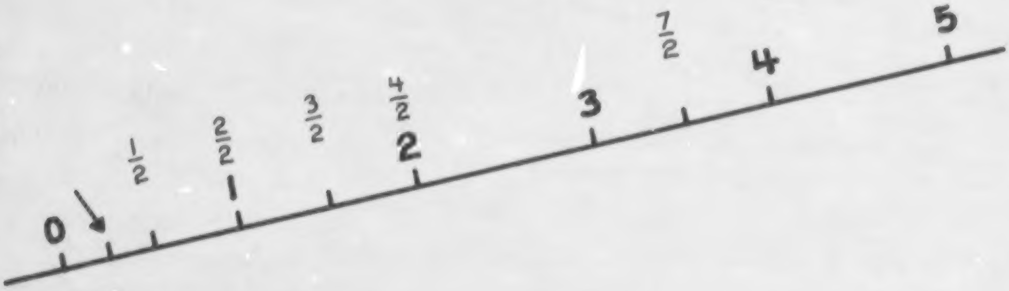
N 17

Continue to label points for D's third, fourth, and seventh steps.



T: Now, look at A. A takes four steps from 0 to 1. Point to the place where A's first step lands.

Invite a student to locate A's first step.



For the benefit of those students who have difficulty locating A's first step, lead the discussion as follows.

T: Who takes larger steps, A or D?

S: D; D only takes two steps while A takes four.

T: How many steps does A take for each step that D takes?

S: A takes two steps for each step D takes.

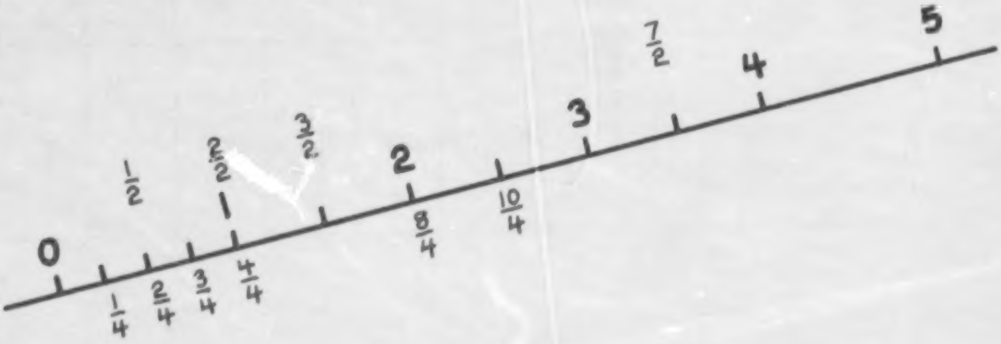
T: Where is A's first step?

S: In the middle, between 0 and $\frac{1}{2}$.

T: How should we label this point?

S: "$\frac{1}{4}$".

Continue by asking students to label points for A's second, third, fourth, eighth, and tenth steps.


0
$\frac{1}{4}$
$\frac{2}{4}$
$\frac{3}{4}$
$\frac{4}{4}$
$\frac{1}{2}$
$\frac{2}{2}$
1
$\frac{3}{2}$
2
$\frac{8}{4}$
$\frac{10}{4}$
3
$\frac{7}{2}$
4
5

Exercise 3

Add this row of information to your table.

Child	A	B	C	D	E	F	G	H	J
Number of steps to reach 1	4	12	15	2	5	10	9	3	99
First step lands on (step size)	$\frac{1}{4}$			$\frac{1}{2}$					

N 17

Invite students to complete the table by writing a fraction for each child's first step. A completed table follows.

Child	A	B	C	D	E	F	G	H	J
Number of steps to reach 1	4	12	15	2	5	10	9	3	99
First step lands on (step size)	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{15}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{99}$

T (pointing to "$\frac{1}{4}$", "$\frac{1}{12}$", . . . , "$\frac{1}{99}$"): Which of these numbers is the largest?

S: $\frac{1}{2}$.

Write "$\frac{1}{2}$" on the board.

T: Which is the smallest?

S: $\frac{1}{99}$.

Write "$\frac{1}{99}$" on the board to the right of "$\frac{1}{2}$" and with a large space between them.

$$\frac{1}{2} \qquad\qquad > \qquad\qquad \frac{1}{99}$$

T: On your paper, arrange these numbers ($\frac{1}{4}$, $\frac{1}{12}$, . . . , $\frac{1}{99}$) from the largest to the smallest.

Let the students work independently on this problem. As the class finishes the work, ask students to list the fractions from the largest to smallest on the board.

$$\frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \frac{1}{5} > \frac{1}{9} > \frac{1}{10} > \frac{1}{12} > \frac{1}{15} > \frac{1}{99}$$

Exercise 4

T: The children sometimes play a game with this rule: to score a point, you must step on the $\frac{1}{2}$ mark. Which children will score a point?

S: D will.

S: A will.

T: How many steps must A take to go from 0 to 1? (Four) How many steps must A take to reach $\frac{1}{2}$?

S: Two steps.

Write this information on the board indicating that you are abstracting from the discussion.

$$\frac{1}{2} = \frac{2}{4}$$

T: Look at the table. Which other children can land on $\frac{1}{2}$ and how many steps will each child take?

N 17

S: B will land on $\frac{1}{2}$ after taking six steps.

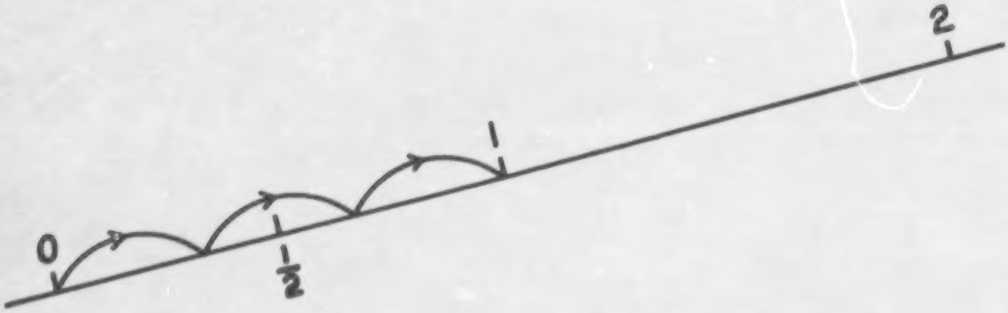
Include this information on the board.

$$\frac{1}{2} = \frac{2}{4} = \frac{6}{12}$$

S: F will land on $\frac{1}{2}$ after taking five steps.

$$\frac{1}{2} = \frac{2}{4} = \frac{6}{12} = \frac{5}{10}$$

Use the number line to settle any disputes. For example, to show that $\frac{1}{2}$ has no "thirds-name" invite a student to trace the path of a child (H) who takes three steps from 0 to 1.



T: What are some other names for $\frac{1}{2}$ that are not suggested by the table?

Let the students suggest many other names for $\frac{1}{2}$.

$$\frac{1}{2} = \frac{2}{4} = \frac{6}{12} = \frac{5}{10} = \frac{10}{20} = \frac{50}{100} = \frac{200}{400}$$

T: Do you notice anything interesting about the names for $\frac{1}{2}$?

S: The bottom number is always the double of the top number.

S: The bottom number is always even.

Accept any reasonable comments.

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N 21

3.6 Sasquatch #2

CAPSULE LESSON SUMMARY

Review the story of Sasquatch and the game (from Lesson N3.5) the children played using the footprints. Locate various rational numbers on the number line and observe, for example, that $\frac{2}{6} = \frac{1}{3}$ and $\frac{12}{6} = \frac{6}{3} = 2$. In this situation, compare rational numbers and determine which numbers can occur as denominators of fractions for $\frac{1}{3}$ and for $\frac{1}{4}$. For example, observe that $\frac{1}{4} < \frac{1}{3}$ but $\frac{2}{4} > \frac{1}{3}$, and $\frac{1}{3} = \frac{8}{24} = \frac{2}{6} = \frac{4}{12} = \ldots$, and $\frac{1}{4} = \frac{6}{24} = \frac{3}{12} = \frac{5}{20} = \ldots$. Determine which numbers occur as denominators of fractions for both $\frac{1}{3}$ and $\frac{1}{4}$.

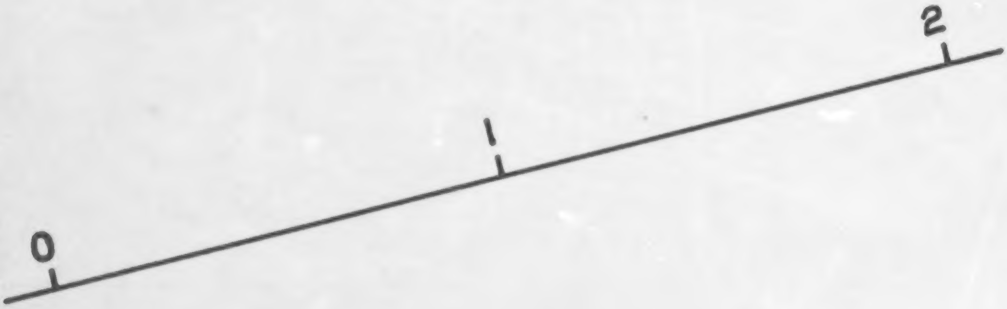
MATERIALS

Teacher: Colored chalk; chalkboard compass (optional)
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Draw this part of the number line on the board.



T: Do you remember the story of Sasquatch?

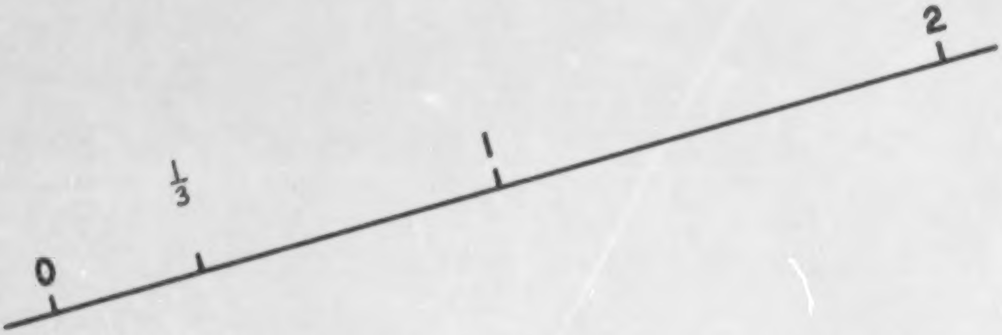
Let the students recall as much of the legend of Sasquatch as they can. Include in the review an explanation of the game the children played with Big Foot's tracks.

S: The children play a game using the tracks. They start walking from one of the footprints (at 0). Each child is careful to take steps of the same length. Each time a child steps on one of the Big Foot tracks that child scores a point.

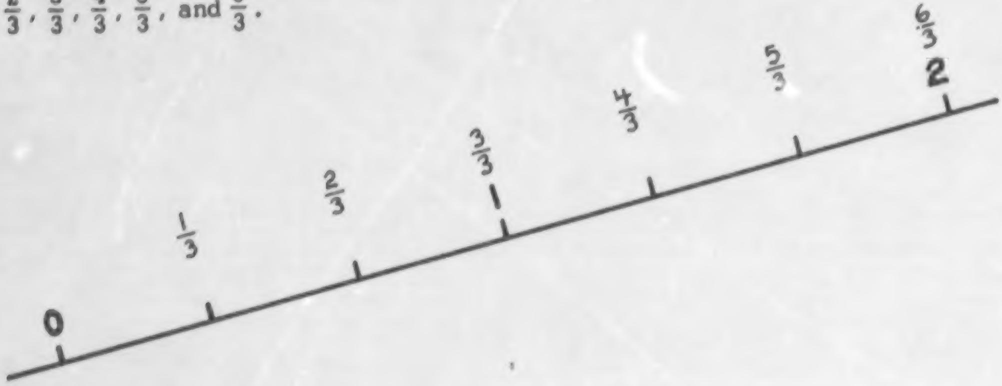
If you like, invite students to "play" the game as they did in Lesson 3.5 Sasquatch #1.

T: One of the children, E, who plays the game takes three steps from 0 to 1. Who can show on the number line where this child's first step lands?

Ask a student to locate and label the point.



Continue in this manner by asking the students to locate and label points for $\frac{2}{3}$, $\frac{3}{3}$, $\frac{4}{3}$, $\frac{5}{3}$, and $\frac{6}{3}$.



T: Another child, D, takes six steps from 0 to 1. Is D's step length larger or smaller than E's step-length?

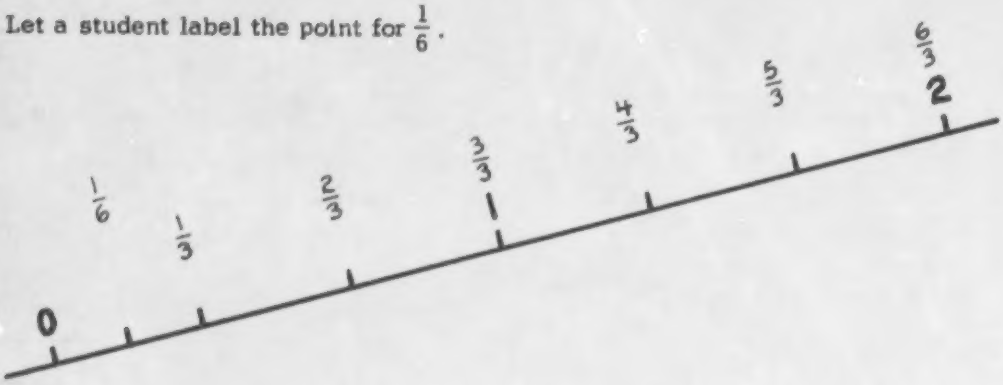
N 21

S: Smaller; it takes two of D's steps to equal one of E's steps.

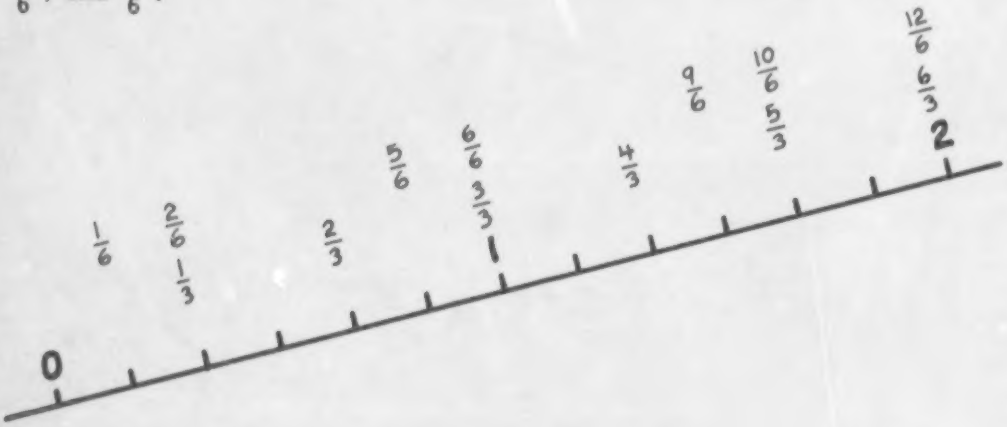
T: Good. Where on the number line is D's first step?

S: Half-way between 0 and $\frac{1}{3}$.

Let a student label the point for $\frac{1}{6}$.


0
$\frac{1}{6}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{3}{3}$
1
$\frac{4}{3}$
$\frac{5}{3}$
$\frac{6}{3}$
2

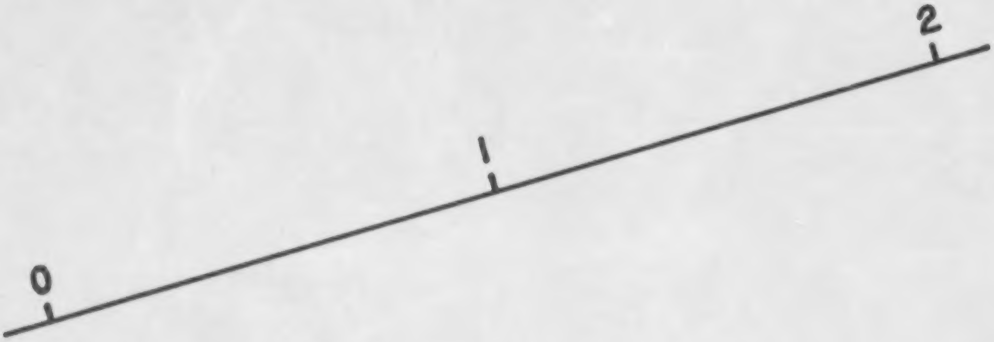
Continue by asking students to locate and label points for $\frac{2}{6}$, $\frac{5}{6}$, $\frac{6}{6}$, $\frac{9}{6}$, $\frac{10}{6}$, and $\frac{12}{6}$.


0
$\frac{1}{6}$
$\frac{2}{6}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{5}{6}$
$\frac{6}{6}$
$\frac{3}{3}$
1
$\frac{4}{3}$
$\frac{9}{6}$
$\frac{10}{6}$
$\frac{5}{3}$
$\frac{12}{6}$
$\frac{6}{3}$
2

As this activity progresses, students should note that $\frac{2}{6} = \frac{1}{3}$, $\frac{6}{6} = \frac{3}{3} = 1$, $\frac{10}{6} = \frac{5}{3}$, and $\frac{12}{6} = \frac{6}{3} = 2$.

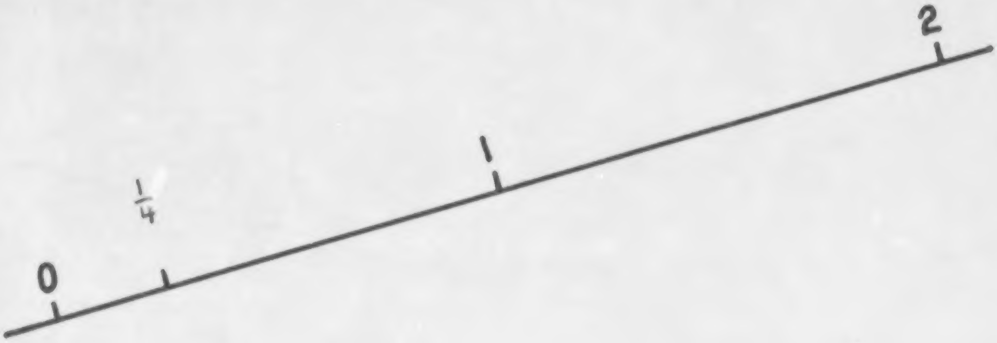
Exercise 2

Redraw this portion of the number line and ask the students to copy it on their papers.



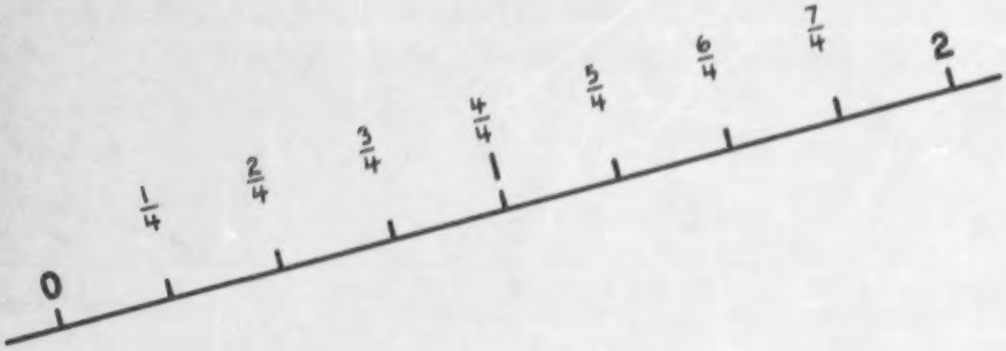
T: Child B takes four steps from 0 to 1. Locate the point where B's first step lands and label the point on your number line.

Let the class work on this problem independently for a while. Ask a student to locate $\frac{1}{4}$ at the board.



Instruct your students to locate B's second, third, fourth, fifth, sixth, and seventh steps on their paper; then at the board.

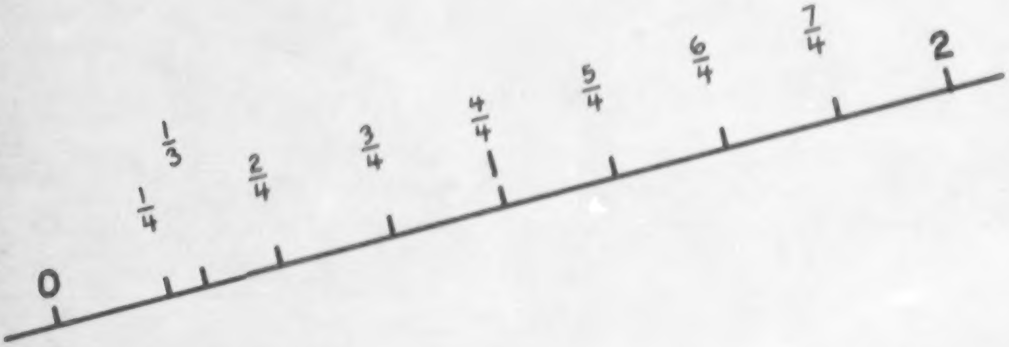
N 21


0
$\frac{1}{4}$
$\frac{2}{4}$
$\frac{3}{4}$
$\frac{4}{4}$
1
$\frac{5}{4}$
$\frac{6}{4}$
$\frac{7}{4}$
2

T: Child E takes three steps from 0 to 1. Will E's first step be to the right or to the left of $\frac{1}{4}$?

S: $\frac{1}{3}$ will be to the right of $\frac{1}{4}$; $\frac{1}{3}$ is larger than $\frac{1}{4}$.

Invite a student to locate, approximately, $\frac{1}{3}$ on the number line. One way to do this is to give the student a chalkboard compass and let the student use the compass to divide the segment from 0 to 1 into three pieces of equal length.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$\frac{1}{3}$
$\frac{2}{4}$
$\frac{3}{4}$
$\frac{4}{4}$
1
$\frac{5}{4}$
$\frac{6}{4}$
$\frac{7}{4}$
2

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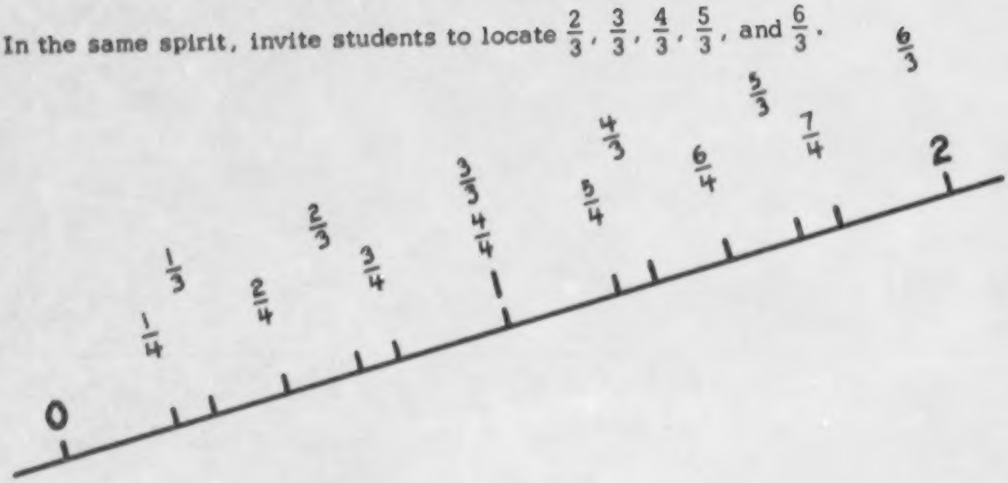
In the same spirit, invite students to locate $\frac{2}{3}$, $\frac{3}{3}$, $\frac{4}{3}$, $\frac{5}{3}$, and $\frac{6}{3}$.



0
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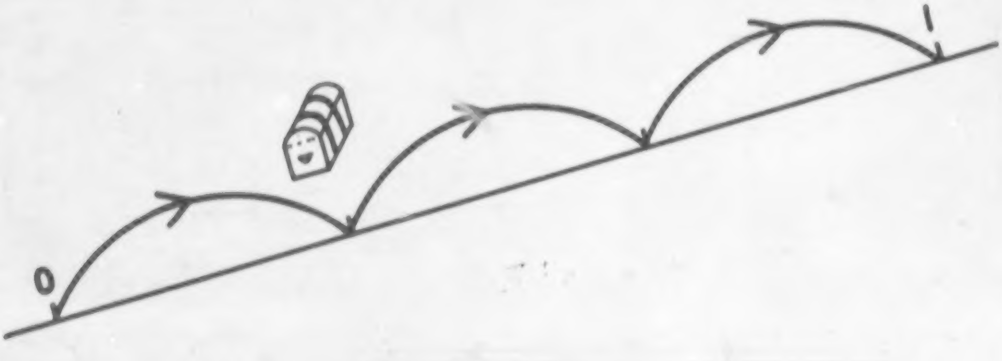
$\frac{1}{3}$
$\frac{2}{4}$
$\frac{2}{3}$
$\frac{3}{4}$
$\frac{3}{3}$
$\frac{4}{4}$
1
$\frac{5}{4}$
$\frac{4}{3}$
$\frac{6}{4}$
$\frac{5}{3}$
$\frac{7}{4}$
$\frac{6}{3}$
2

Exercise 3

Draw this table on the board and draw part of the number line as illustrated below.

Child	A	B	C	D	E	F	G	H	J	K
First step lands on (step size)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{15}$



N 21

T: One day some children discovered that a treasure was buried in the snow-path between the Sasquatch tracks.

(Point to the illustrated treasure chest.) Where is the treasure located?

S: At $\frac{1}{3}$.

T (pointing to the second row of the table): The children knew their step-size and decided that those children whose steps land on the treasure would share it. Which children should share the treasure? Explain your answers.

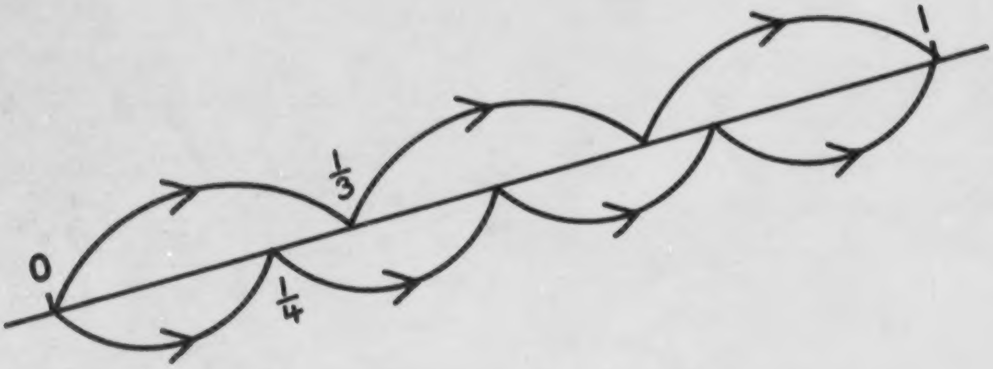
S: A would miss the treasure, because A's first step lands on $\frac{1}{2}$ which is larger than $\frac{1}{3}$.

S: D would land on the treasure, because D takes two steps and lands on $\frac{1}{3}$.

Summarize each student's explanation in your table. When complete your table should look like the one below.

Child	A	B	C	D	E	F	G	H	J	K
First step lands on (step size)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{15}$
Treasure at $\frac{1}{3}$	X	X	$\frac{8}{24}$	$\frac{2}{6}$	$\frac{1}{3}$	$\frac{4}{12}$	X	$\frac{6}{18}$	$\frac{3}{9}$	$\frac{5}{15}$

Continue the story by introducing a buried treasure at $\frac{1}{4}$.



Ask which children should share the second treasure. A completed table is shown below.

Child	A	B	C	D	E	F	G	H	J	K
First step lands on (step size)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{24}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{15}$
Treasure at $\frac{1}{3}$	×	×	$\frac{8}{24}$	$\frac{2}{6}$	$\frac{1}{3}$	$\frac{4}{12}$	×	$\frac{6}{18}$	$\frac{3}{9}$	$\frac{5}{15}$
Treasure at $\frac{1}{4}$	×	$\frac{1}{4}$	$\frac{6}{24}$	×	×	$\frac{3}{12}$	$\frac{5}{20}$	×	×	×

T (pointing to the third row): Do you notice anything interesting about these fractions?

S: Each fraction is another name for $\frac{1}{3}$.

The class should also conclude that the fractions in the fourth row are names for $\frac{1}{4}$.

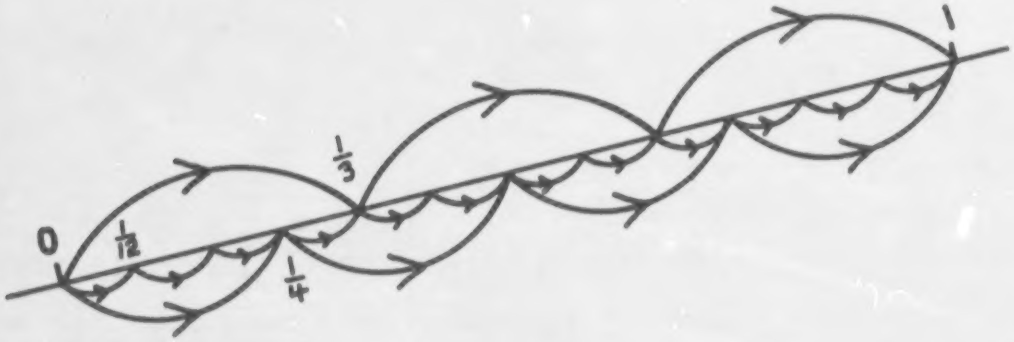
N 21

T: Which children receive a share of both treasures?

S: F and C.

T: F's step-size is $\frac{1}{12}$. Can you locate $\frac{1}{12}$ on the number line?

Invite a student to locate, approximately, $\frac{1}{12}$. A chalkboard compass could be used as an aid in locating $\frac{1}{12}$. Once $\frac{1}{12}$ has been located, locate $\frac{2}{12}$, $\frac{3}{12}$, etc. on the number line.



S: F lands on $\frac{1}{4}$ after three steps and on $\frac{1}{3}$ after four steps.

T: What about C? C's step-size is $\frac{1}{24}$. Locate $\frac{1}{24}$ on the number line.

S: $\frac{1}{24}$ is half-way between 0 and $\frac{1}{12}$.

A student should indicate the point for $\frac{1}{24}$.

S: C takes six steps to land on $\frac{1}{4}$ and eight steps to land on $\frac{1}{3}$.

T: Good. Can you think of some other step-sizes for children (not those in our table) who would receive a share of both treasures if they were playing?

S: $\frac{1}{36}$.

List the students' suggestions on the board.

$$\frac{1}{12}, \frac{1}{24}, \frac{1}{36}, \frac{1}{48}, \frac{1}{72}, \frac{1}{60}, \frac{1}{600}, \ldots$$

T: Do you notice anything interesting about these fractions?

S: The bottom numbers are multiples of 3 and multiples of 4.

S: The bottom numbers are multiples of 12.

Accept any reasonable explanation.

N 22

3.7 Fractions and Decimal Names #1

CAPSULE LESSON SUMMARY

Solve hand-calculator sentences both using and not using memory keys. Insert parentheses in related number sentences to indicate how the calculations are done on the calculator. Relate fractions with division and find decimal names on the calculator. Then use the calculator to do calculations such as "$\frac{5}{2} + \frac{4}{10}$".

MATERIALS

Teacher: Colored chalk; hand-calculator
Student : Hand-calculator (see note)

DESCRIPTION OF LESSON

NOTE: This lesson explores the use of the memory keys that are found on many calculators. The lesson description makes use of calculators with an add to memory key ($\boxed{\text{M+}}$) and a subtract from memory key ($\boxed{\text{M−}}$). To accommodate those classes that will use calculators with only a memory store key ($\boxed{\text{M}}$ or $\boxed{\text{STO}}$) and a memory recall key ($\boxed{\text{MRC}}$ or $\boxed{\text{RCL}}$), the hand-calculator expressions are written in such a way that the $\boxed{\text{M+}}$ functions exactly as a $\boxed{\text{M}}$ key. For a more detailed commentary on the use of hand-calculators, see Section 2(f) of the General Introduction.

Provide each student with a calculator having the memory feature.

Exercise 1

T: Turn on your calculators, hide the display, and press the buttons for the problem that I write on the board.

Write this problem on the board, asking students to press the appropriate key as you write each symbol.

$$4 \times 5 + 10 \times 8 = \boxed{}$$

T: Which number should be on the display?

S: 240.

Write "240" in the box.

T: Can you put parentheses in this equation to show the order in which the operations are performed?

Invite a student to insert the parentheses.

$$((4 \times 5) + 10) \times 8 = \boxed{240}$$

S: First calculate "4 × 5", so we need these parentheses (in red). Then add 10 to 4 × 5, so we need these parentheses (in black). Then multiply that number (30) by 8 to get 240.

T: Let's do another calculation. Hide the display and try to predict which number should be on the display.

Write this information on the board.

$$4 \times 5 = \text{M+}\ 10 \times 8\ + \text{MRC} =$$

S: 100.

T: Can you explain how the calculation is performed?

S: 4 × 5 = 20 and 20 was put in the memory. Then "10 × 8" was calculated; 20 was recalled from the memory and added to 80 to give 100.

Write an equation under the hand-calculator sentences leaving spaces to suggest the order in which operations are performed.

$$4 \times 5 = \text{M+} \; 10 \times 8 + \text{MRC} =$$

$$4 \times 5 + 10 \times 8 = \boxed{100}$$

T: Can we put parentheses in this number story to show how the calculation is done?

$$(4 \times 5) + (10 \times 8) = \boxed{100}$$

Ask the students to clear the display and the memory of their hand-calculators.

NOTE: Although there are other ways to clear the display and memory, an easy way is just to turn the calculator off and then on again.

Repeat the above activity using the problems below. For each example, the students should add parentheses to indicate how the calculations are performed.

$$6 \times 5 - 4 \times 3 = \boxed{}$$

Answer: $((6 \times 5) - 4) \times 3 = \boxed{78}$

$$4 \times 3 = \text{M+} \; 6 \times 5 - \text{MRC} =$$

Answer: $(6 \times 5) - (4 \times 3) = \boxed{18}$

Exercise 2

T: How can we display $\frac{1}{2}$ on the calculator without using the $\boxed{\cdot}$ key?

Let the students consider this problem for a short while.

S: Press $\boxed{1}\ \boxed{\div}\ \boxed{2}\ \boxed{=}$.

Write this equality on the board.

$$0.5 = 1 \div 2 = \frac{1}{2}$$

T: How can we display $\frac{2}{4}$?

S: Press $\boxed{2}\ \boxed{\div}\ \boxed{4}\ \boxed{=}$.

S: $\frac{2}{4} = 0.5 = \frac{1}{2}$.

T: Hide the display and use the calculator to find a decimal name for $\frac{3}{4}$. Which keys do you press?

S: Press $\boxed{3}\ \boxed{\div}\ \boxed{4}\ \boxed{=}$. 0.75 is on the display; $\frac{3}{4} = 0.75$.

T: Good. Can you suggest some other fractional names for 0.75?

As the students provide other names, write them on the board.

$$0.75 = \frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{75}{100} = \ldots$$

Encourage the class to check some of the fractional names for 0.75 by performing division calculations on their calculators.

T: Hide the display and use your calculator to find a decimal name for $\frac{3}{10}$. Which keys (other than $\boxed{\cdot}$) do you press?

S: Press $\boxed{3}\ \boxed{\div}\ \boxed{10}\ \boxed{=}$. 0.3 is on the display; $\frac{3}{10} = 3 \div 10 = 0.3$.

Repeat this activity using $\frac{4}{100}$; $\frac{35}{10}$; $\frac{495}{100}$; and $\frac{3}{1,000}$.

Exercise 3

This exercise should develop in a manner similar to that of Exercise 1.

Write the problem below on the board and ask the students to perform the calculation on their calculators.

$$1 \div 2 + 1 \div 2 = \boxed{}$$

Let a student insert parentheses and explain how 0.75 is obtained.

$$((1 \div 2) + 1) \div 2 = 0.75$$

Continue the activity by asking the students to perform a calculation using memory keys.

$$1 \div 2 = \mathrm{M+}\ \ 1 \div 2 + \mathrm{MRC} =$$

$$(1 \div 2) + (1 \div 2) = \boxed{1}$$

T: Do you know another way to write this calculation?

S: $1 \div 2$ is $\frac{1}{2}$, so "$\frac{1}{2} + \frac{1}{2} = 1$".

Ask the class to clear the display and memory of their calculators. Follow a similar procedure with the two problems below.

$$5 \div 2 \quad + \quad 4 \div 10 \quad = \square$$

$$5 \div 2 = \mathrm{M}{+}\ 4 \div 10 + \mathrm{MRC} =$$

The class should conclude that the first calculation is for 0.65 while the second is for 2.9.

$$((5 \div 2) + 4) \div 10 = \boxed{0.65}$$

$$(5 \div 2) + (4 \div 10) = \tfrac{5}{2} + \tfrac{4}{10} = \boxed{2.9}$$

N 24

3.8 Addition of Rational Numbers #1

CAPSULE LESSON SUMMARY

Find many fractional names for rational numbers such as $\frac{1}{2}$ and $\frac{2}{3}$. Solve addition problems such as "$\frac{1}{2} + \frac{3}{4} = ?$" and "$\frac{2}{3} + \frac{3}{4} = ?$" using appropriate partitions of regions.

MATERIALS

Teacher: Colored chalk
Student: Colored pencils; Worksheets N41, N42

DESCRIPTION OF LESSON

Exercise 1

T: Can you suggest some other fractions for $\frac{1}{2}$?

As students suggest names for $\frac{1}{2}$ write them on the board in a string of equalities.

$$\frac{1}{2} = \frac{2}{4} = \frac{6}{12} = \frac{7}{14} = \cdots$$

After several names for $\frac{1}{2}$ have been suggested, ask the class to complete several problems like those suggested below.

$$\frac{1}{2} = \frac{2}{4} = \frac{6}{12} = \frac{7}{14} = \frac{\square}{8} = \frac{9}{\square} = \frac{\square}{16} = \frac{12}{\square}$$

S: $\frac{1}{2} = \frac{4}{8}$.

S: $\frac{1}{2} = \frac{9}{18}$.

S: $\frac{1}{2} = \frac{8}{16}$.

S: $\frac{1}{2} = \frac{12}{24}$.

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Repeat the above activity by eliciting fractions for $\frac{2}{3}$. If you prefer, instruct the students to write the names they find on their papers. List some of the suggested names for $\frac{2}{3}$ on the board, and ask the class to solve several problems.

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{\boxed{9}} = \frac{\boxed{12}}{18} = \frac{8}{\boxed{12}} = \frac{\boxed{14}}{21} = \frac{10}{\boxed{15}}$$

Exercise 2

Write this problem on the board.

$$\frac{1}{2} + \frac{3}{4} =$$

T: Solve this problem on your paper.

Walk around and observe the students' answers. Below the problem draw two regions the same size.

T (pointing to the region at the left): Shade one-half of this region.

Invite a student to do the shading and to write "$\frac{1}{2}$" below the region.

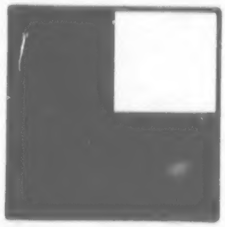
N 24

$\frac{1}{2}$

T: Shade three-fourths of the other region.

Invite a student to do the shading and to write "$\frac{3}{4}$" below the region.

$\frac{1}{2}$

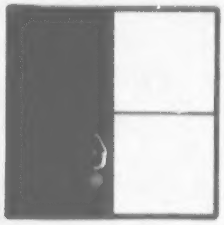


$\frac{3}{4}$

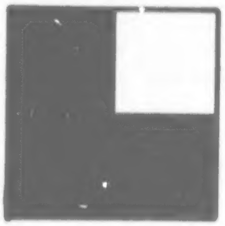
T: Does the picture suggest a "fourths name" for $\frac{1}{2}$?

S: Yes, $\frac{1}{2} = \frac{2}{4}$.

Put this information on the board.



$\frac{1}{2} = \frac{2}{4}$



$\frac{3}{4}$

T: How can we use the drawing to help us calculate "$\frac{1}{2} + \frac{3}{4}$"?

S: $\frac{1}{2} + \frac{3}{4} = \frac{2}{4} + \frac{3}{4} = \frac{5}{4}$, because altogether there are five pieces shaded red and each piece is one-fourth of a region.

Summarize the student's explanation.

$$\frac{1}{2} + \frac{3}{4} = \frac{2}{4} + \frac{3}{4} = \frac{5}{4}$$

T: Can you suggest another name for $\frac{5}{4}$?

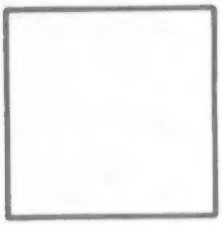
S: "$1\frac{1}{4}$".

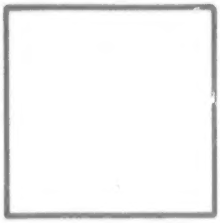
S: "$\frac{10}{8}$" or "$\frac{20}{16}$" and so on.

Exercise 3

Write this calculation on the board. Draw two regions the same size below the calculation.

$$\frac{2}{3} + \frac{3}{4} =$$

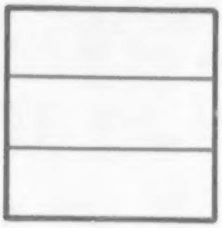


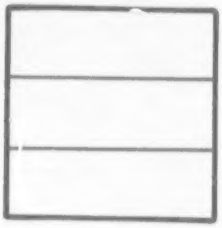


N 24

T: Nabu invented a special method to do addition problems with fractions. He divided each region into three rows and four columns.

Invite a student to perform the partition of both regions into three rows.





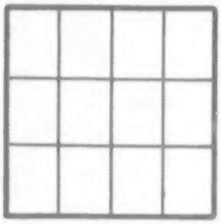
T: Why do you think he divided the regions into three rows?

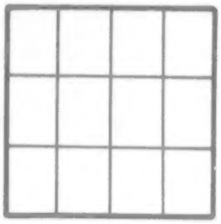
S: $\frac{2}{3}$ is one of the numbers being added and we divided the regions into thirds.

T: Good. Nabu divided each region into four columns. Why?

S: Because the other number is $\frac{3}{4}$.

If explanations similar to these are not forthcoming, make the suggestions yourself. Then invite a student to make the column partition of the regions.



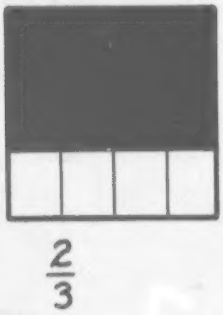


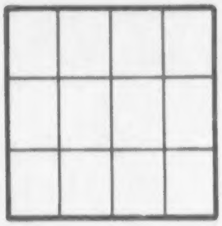
T: How many little pieces in each region?

S: 12; 3 x 4 = 12.

T (pointing to the region at the left): Let's shade two-thirds of this region.

Invite a student to do the shading and to write "$\frac{2}{3}$" below the figure.





T: What fractional part of the region is each little piece?

S: One-twelfth.

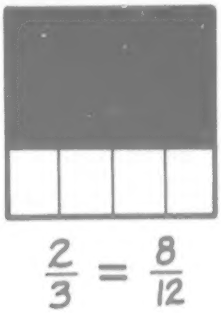
T: How many little pieces are shaded?

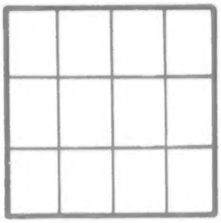
S: 8.

T: Does this picture suggest another name for $\frac{2}{3}$?

S: "$\frac{8}{12}$".

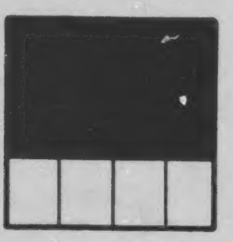
Add this information to your drawing.



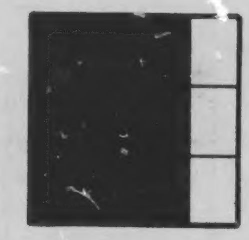


N 24

In a similar manner, direct the class to shade three-fourths of the region at the right and to conclude that $\frac{3}{4} = \frac{9}{12}$.



$$\frac{2}{3} = \frac{8}{12}$$



$$\frac{3}{4} = \frac{9}{12}$$

T: Which is larger, $\frac{2}{3}$ or $\frac{3}{4}$? Why?

S: $\frac{3}{4}$ is larger; for $\frac{3}{4}$ nine pieces are shaded compared to eight pieces for $\frac{2}{3}$.

T: Look at both of the regions carefully. How many little pieces in all are shaded?

S: 17.

T: Remember, we said each little piece was one-twelfth of the square, so what fraction can we give to the shaded regions?

S: $\frac{\text{"}17\text{"}}{12}$.

Direct the students to complete the calculation.

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}$$

T: Is $\frac{17}{12}$ larger or smaller than 1?

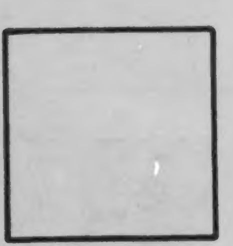
S: Larger than 1; $\frac{12}{12} = 1$.

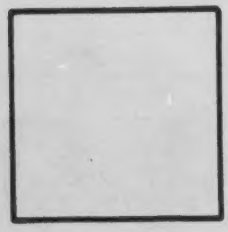
S: $\frac{17}{12} = 1\frac{5}{12}$.

Exercise 4

Write this calculation on the board. Draw two regions the same size below the calculation.

$$\frac{1}{2} + \frac{3}{5} =$$





T: Let's use Nabu's method to calculate "$\frac{1}{2} + \frac{3}{5}$". Copy the information on the board onto your paper.

How many rows and how many columns should we divide each region into?

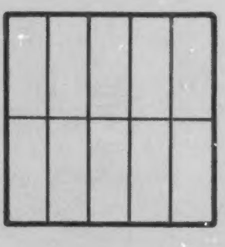
S: Two rows and five columns are needed, because halves and fifths are being added.

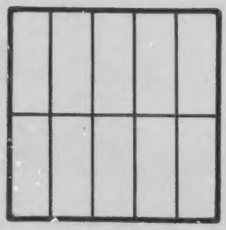
T: Very good. On your paper divide each region into two rows and five columns.

NOTE: Five rows and two columns could also be used.

N 24

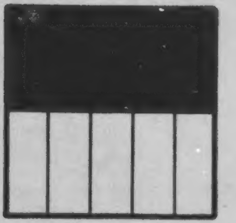
Let the students work independently for a short while. Then ask students to partition both regions in the board drawing.



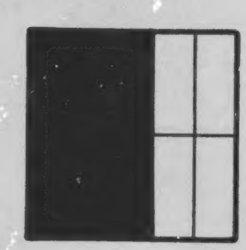


Continue this activity in the same spirit as Exercise 3. However, as each step of the problem is solved, first let the students work independently and then check collectively. The class should conclude that $\frac{1}{2} + \frac{3}{5} = \frac{5}{10} + \frac{6}{10} = \frac{11}{10}$.

$$\frac{1}{2} + \frac{3}{5} = \frac{5}{10} + \frac{6}{10} = \frac{11}{10}$$



$$\frac{1}{2} = \frac{5}{10}$$



$$\frac{3}{5} = \frac{6}{10}$$

For this problem, call the class' attention to the connection between decimal names and fractions.

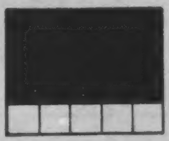
$$\frac{1}{2} + \frac{3}{5} = \frac{5}{10} + \frac{6}{10} = 0.5 + 0.6 = 1.1 = \frac{11}{10}$$

Worksheets N41 and N42 are available.

Name ______________________

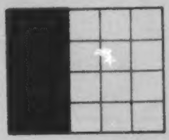
N 41 *

Color three-fourths of this region and fill in the box.



$$\frac{3}{4} = \frac{\boxed{15}}{20}$$

Color two-fifths of this region and fill in the box.



$$\frac{2}{5} = \frac{\boxed{8}}{20}$$

Complete:

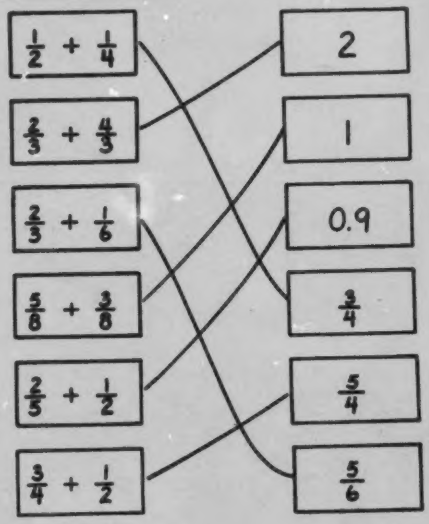
$$\frac{3}{4} + \frac{2}{5} = \frac{\boxed{15}}{20} + \frac{\boxed{8}}{20} = \frac{23}{20}$$

(Other correct names for $\frac{23}{20}$ may be given; e.g., $\frac{23}{20} = 1\,\frac{3}{20}$.)

Name ______________________

N 42 **

Match the red and the blue boxes.

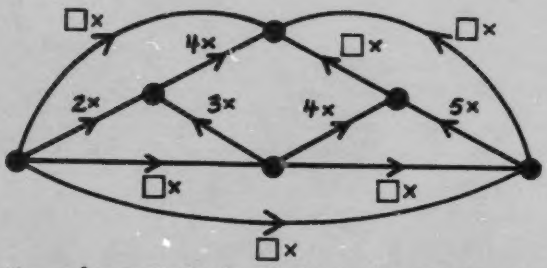


$\frac{1}{2} + \frac{1}{4}$
$\frac{2}{3} + \frac{4}{3}$
$\frac{2}{3} + \frac{1}{6}$
$\frac{5}{8} + \frac{3}{8}$
$\frac{2}{5} + \frac{1}{2}$
$\frac{3}{4} + \frac{1}{2}$
2
1
0.9
$\frac{3}{4}$
$\frac{5}{4}$
$\frac{5}{6}$

3.9 Multiplication of Rational Numbers #2

CAPSULE LESSON SUMMARY

Review composition of multiplication functions to label arrows in pictures such as:



Write fractions for certain decimals and multiply decimals by first finding a fraction for each decimal number.

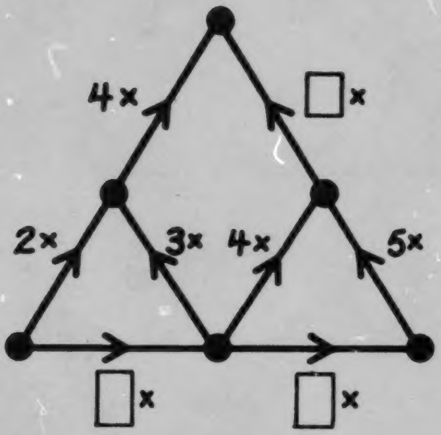
MATERIALS

Teacher: Colored chalk

Student: Hand-calculator; paper; colored pencils; Worksheets N43, N44, N45, N46

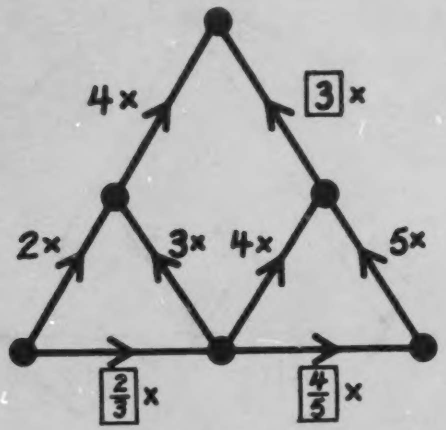
DESCRIPTION OF LESSON

Draw this arrow picture on the board.

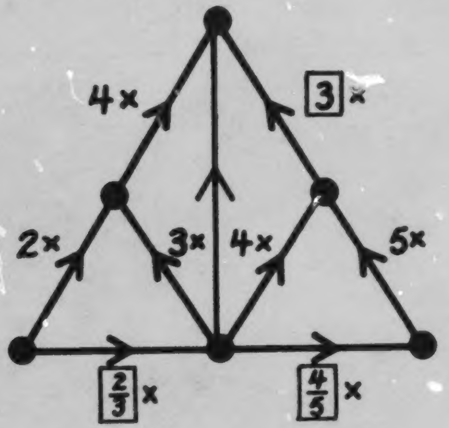


197

Invite students to fill in the box for any arrow they wish. Answers are provided below.



If there are some students who have difficulty labeling the 3 x arrow, add this blue supplementary arrow to your picture.



T: What could this blue arrow be for?

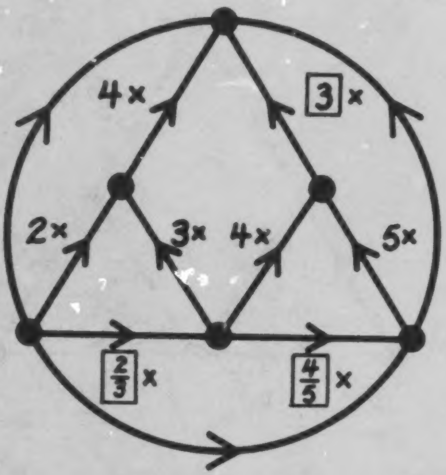
S: 12 x.

T: Good. The blue arrow is for 3 x followed by 4 x or 12 x. (Point to the blank box.) What could this red arrow be for?

S: 3 x, because 4 x followed by 3 x is also 12 x.

N 30

Erase the 12 x arrow and add these three blue arrows to your picture. Ask the students to label them in the order suggested in the following dialogue.

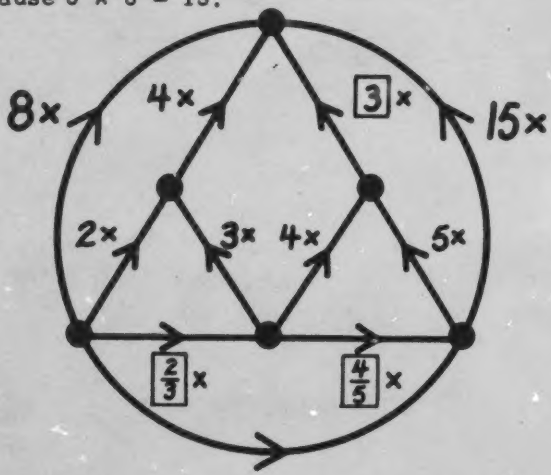

4x
3 x
2x
3x
4x
5x
2/3 x
4/5 x

T: Look at the blue arrow to the left. It is for 2 x followed by 4 x. What could this blue arrow be for?

S: 8 x, because 2 x 4 = 8.

T: Now look at the blue arrow to the right. It is for 5 x followed by 3 x. What could this blue arrow be for?

S: 15 x, because 5 x 3 = 15.


8x
4x
3 x
15x
2x
3x
4x
5x
2/3 x
4/5 x

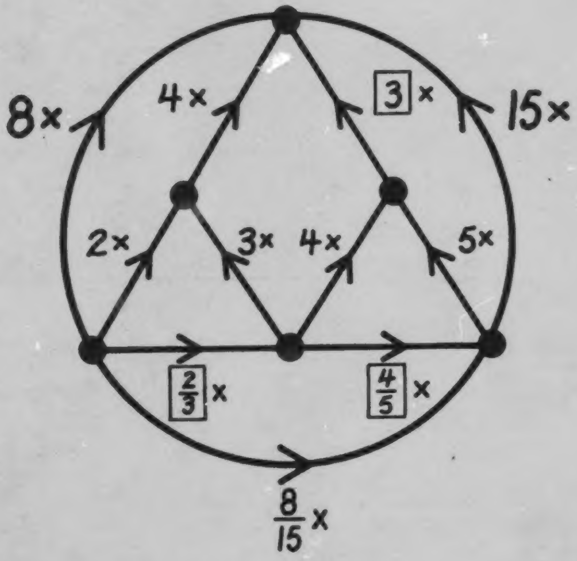
199

T: Look at the blue arrow at the bottom. What could this blue arrow be for? Use the other two blue arrows to help you.

Let the class have a few minutes to consider the situation.

S: $\frac{8}{15}$x; it is 8 x followed by "against" 15 x.

T (tracing "with" or "against" the appropriate arrows in turn): 8 x followed by "against" 15 x is $\frac{8}{15}$x.



T: Do you see another way to look at this arrow as a composition of arrows?

S: It is $\frac{2}{3}$x followed by $\frac{4}{5}$x.

Write this calculation on the board.

$$\frac{2}{3} \times \frac{4}{5} =$$

N 30

T: How can we calculate "$\frac{2}{3} \times \frac{4}{5}$"?

Summarize the students comments by adding this information to the calculation.

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$$

T: $\frac{2}{3}\times$ followed by $\frac{4}{5}\times$ is $\frac{8}{15}\times$.

Exercise 2

Provide a calculator for each student.

T: Hide the display and press $\boxed{3}\ \boxed{\div}\ \boxed{10}\ \boxed{=}$. Which number is on the display?

S: 0.3.

T: Look at the display to check your answer.

Write this calculation on the board.

$$3 \div 10 = 0.3$$

T: Do you know a fraction for 0.3?

S: "$\frac{3}{10}$".

Add this information to the calculation on the board.

$$3 \div 10 = 0.3 = \frac{3}{10}$$

Continue this activity using the division problems given below. Each time ask the students to predict the decimal name, then to give a fractional name (for this activity a fraction whose denominator is 10 or 100 or 1,000 or . . .).

$$7 \div 10 = \boxed{0.7 = \frac{7}{10}}$$

$$15 \div 10 = \boxed{1.5 = \frac{15}{10}}$$

$$40 \div 10 = \boxed{4.0 = \frac{40}{10}}$$

$$6 \div 100 = \boxed{0.06 = \frac{6}{100}}$$

$$25 \div 100 = \boxed{0.25 = \frac{25}{100}}$$

$$325 \div 100 = \boxed{3.25 = \frac{325}{100}}$$

Exercise 3

Write this calculation on the board.

$$0.2 \times 0.3 =$$

T: Calculate "0.2 × 0.3"? Do not use your calculators.

Write the students' suggestions (correct or incorrect) on the board. A common mistake is to answer 0.6.

T: Several suggestions have been made. How did you perform this calculation?

S: I used fractions.

If this method is not suggested, make it yourself.

T: Very good. Can you give a fraction for 0.2?

S: "$\frac{2}{10}$".

T: And a fraction for 0.3?

S: "$\frac{3}{10}$".

Write this information on the board.

$$0.2 \times 0.3 = \frac{2}{10} \times \frac{3}{10}$$

T: What number is $\frac{2}{10} \times \frac{3}{10}$?

S: $\frac{6}{100}$.

$$0.2 \times 0.3 = \frac{2}{10} \times \frac{3}{10} = \frac{6}{100}$$

T: Can you give a decimal name for $\frac{6}{100}$?

S: "0.06".

$$0.2 \times 0.3 = \frac{2}{10} \times \frac{3}{10} = \frac{6}{100} = 0.06$$

Contrast the above product by asking the students to copy and solve these problems.

$$2 \times 0.3 = \boxed{\quad\quad}$$

$$0.2 \times 3 = \boxed{\quad\quad}$$

Check the problems by letting students explain the calculations. Answers are given below.

$$2 \times 0.3 = \boxed{0.6}$$

$$0.2 \times 3 = \boxed{0.6}$$

Continue this activity with the problems below. Ask the students to copy and to solve them.

$$3 \times 7 = \boxed{\quad}$$

$$3 \times 0.7 = \boxed{\quad}$$

$$0.3 \times 0.7 = \boxed{\quad}$$

$$30 \times 0.7 = \boxed{\quad}$$

$$0.3 \times 70 = \boxed{\quad}$$

$$30 \times 0.07 = \boxed{\quad}$$

Observe the students' work and give assistance where necessary.

Check the individual work collectively by asking students to explain the calculations at the board. Answers are given in the following illustration.

N 30

$$3 \times 7 = \boxed{21}$$

$$3 \times 0.7 = \boxed{2.1}$$

$$0.3 \times 0.7 = \boxed{0.21}$$

$$30 \times 0.7 = \boxed{21}$$

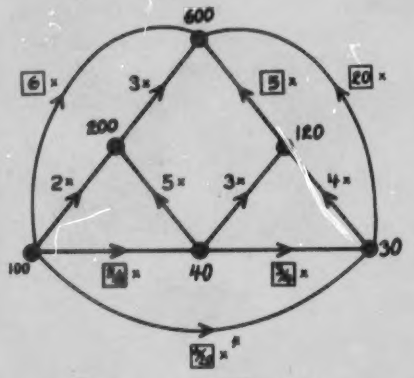
$$0.3 \times 70 = \boxed{21}$$

$$30 \times 0.07 = \boxed{2.1}$$

Worksheets N43, N44, N45, and N46 are available.

Name ____________________ N 43 *

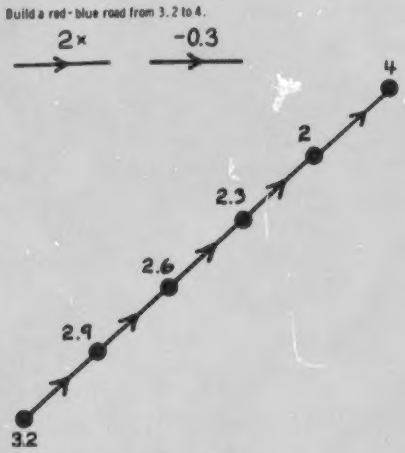
Label the dots and fill in the boxes for the arrows.


600
6 ×
3×
5 ×
20 ×
200
120
2×
5×
3×
4×
100
40
30
×
×
× #

This fractional name is the one that students most likely will give. It is possible that other correct names could be given; e.g., $\frac{6}{20} = \frac{3}{10} = \frac{9}{30} = \frac{12}{40} = \frac{15}{50} = \ldots$

Name ____________________ N 44 **

Build a red-blue road from 3.2 to 4.


2×
-0.3
4
2
2.3
2.6
2.9
3.2

Did you use more than five arrows? No

If your answer is "yes", build a second road using exactly five arrows.

Name ____________________ N 45 ***

First clue

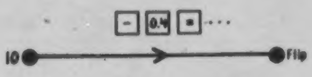
Flip's name can be written using each of these symbols exactly once.

0.2 0.5 2

+ ×) (

Flip could be 0.5, 1.2, 1.4, 2.1, 5.4, or 11.

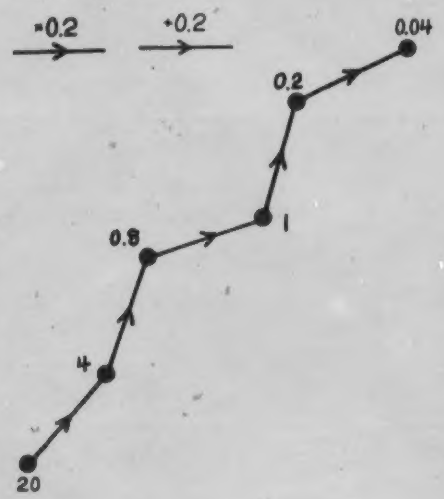
Second clue



Flip is 1.2.

Name ____________________ N 46 ****

Build a red-blue road from 20 to 0.04.


×0.2
+0.2
0.04
0.2
1
0.8
4
20

N 31

3.10 Fractions and Decimal Names #2

CAPSULE LESSON SUMMARY

Find the numbers which can be named by inserting parentheses in the expression "7 - 5 x 2 - 2". Solve hand-calculator sentences both using and not using memory keys. Insert parentheses in related number sentences to indicate how the calculations are done on the calculator. Solve some problems such as "$((8 \times 7) - 6) - (9 \times (15 - 9)) = \square$" on the calculator using the memory feature. Use the calculator to do calculations such as "$\frac{3}{2} + \frac{1}{5}$".

MATERIALS

Teacher: Colored chalk; hand-calculator
Student: Hand-calculator; paper; pencil

DESCRIPTION OF LESSON

NOTE: This lesson explores the use of the memory keys that are found on many calculators. The lesson description makes use of calculators with an add to memory key ($\boxed{\text{M+}}$) and a subtract from memory key ($\boxed{\text{M-}}$). To accommodate those classes that will use calculators with only a memory store key ($\boxed{\text{M}}$ or $\boxed{\text{STO}}$) and a memory recall key ($\boxed{\text{MRC}}$ or $\boxed{\text{RCL}}$), the hand-calculator expressions are written in such a way that the $\boxed{\text{M+}}$ functions exactly as a $\boxed{\text{M}}$ key. For a more detailed commentary on the use of hand-calculators, see Section 2(f) of the General Introduction.

Provide each student with a calculator having the memory feature.

Exercise 1

Write this expression on the board.

$$7 - 5 \times 3 - 2$$

T: Zef's signature is written on the board but he forgot to put in the parentheses. Find some numbers that Zef could be.

Let the students work independently on this problem for a few minutes.

When most students have found several possibilities, invite several students to write the names for the numbers on the board. A complete list is provided here.

$$(7-5)\times(3-2)=2$$

$$(7-(5\times 3))-2 = {}^{-}10$$

$$((7-5)\times 3)-2 = 4$$

$$7-((5\times 3)-2) = {}^{-}6$$

$$7-(5\times(3-2)) = 2$$

Exercise 2

T: Turn on your calculators, hide the display, and press the buttons for the problem I write on the board.

Write the problem below on the board, asking students to press the appropriate key as you write each symbol.

$$8\times 5 - 8\times 4 = \boxed{}$$

T: Which number should be on the display?

S: 128.

T: Check your calculators to see. Can you put parentheses in this number story to show the order in which the calculations are performed?

Ask a student to insert the parentheses.

$$((8\times 5)-8)\times 4 = \boxed{128}$$

S: First calculate "8 × 5", so we need these parentheses (in red). Then subtract 8 from 8 × 5, so we need these parentheses (in black). Then multiply that number (32) by 4 to get 128.

T: Let's do another calculation. Hide the display and try to predict which number should be on the display.

Write this information on the board.

$$8\times 4 = \mathrm{M+}\ 8\times 5 - \mathrm{M_{RC}} =$$

S: 8.

T: Can you explain how the calculation is performed?

S: 8 × 4 = 32 and 32 was put in the memory. Then "8 × 5" was calculated; 32 was recalled from the memory and subtracted from 40 to give 8.

Write an equation under the hand-calculator sentence leaving spaces to suggest the order in which operations are performed.

$$\begin{gathered} 8\times 4 = \mathrm{M+}\ 8\times 5 - \mathrm{M_{RC}} = \\ 8\times 5 \;-\; 8\times 4 = \boxed{8} \end{gathered}$$

T: Who can put the parentheses in this number story to show how the calculation is done?

$$(8 \times 5) - (8 \times 4) = 8$$

Ask the students to clear the memory and display of their hand-calculators and write this problem on the board.

$$100 - (7 \times (15 - 8)) = \square$$

T: Solve this problem on your calculator. Try to use the memory keys. Check that your answer is reasonable by doing the calculation mentally.

Let the students work independently for a few minutes. Each student should have an opportunity to explore how a hand-calculator can be used to perform more complicated calculations.

T: Which number goes in the box?

S: 51.

T: How can we use the calculator to do the calculation?

Let the students describe how they used the hand-calculator. As a student explains, direct the rest of the class to follow along by pressing the keys that the student mentions.

One solution using the memory feature is described below.

S: I pressed $\boxed{15}\ \boxed{-}\ \boxed{8}\ \boxed{\times}\ \boxed{7}\ \boxed{=}\ \boxed{\text{M+}}\ \boxed{100}\ \boxed{-}\ \boxed{\text{MRC}}\ \boxed{=}$.

In a similar manner, ask the students to calculate:

$$((8\times 7)-6)-(9\times(15-9))=\square$$

The students should realize that this problem requires the use of the memory keys. For those students who have difficulty, encourage them to perform the calculation mentally, then use the hand-calculator.

T: Which number goes in the box?

S: ⁻4.

T: How did you get ⁻4?

S: I pressed $\boxed{15}\boxed{-}\boxed{9}\boxed{\times}\boxed{9}\boxed{=}\boxed{M+}\boxed{8}\boxed{\times}\boxed{7}\boxed{-}\boxed{6}\boxed{-}\boxed{MRC}\boxed{=}$.

Other explanations are possible. Ask the class to check a students' method on the calculator.

Exercise 3

T: Clear the display and memory of your calculator. Hide the display and press the buttons for the problem I write on the board.

Write this problem on the board.

$$1\div 2+3\div 2=\square$$

T: Which number should be on the display?

S: 1.75.

T: Check your calculators to see. Can you put parentheses in this equation to show the order in which the operations are performed?

Invite a student to insert the parentheses.

$$((1 \div 2) + 3) \div 2 = \boxed{1.75}$$

Continue the activity by asking the students to follow this expression on their calculators and then discuss what calculation is performed.

$$1 \div 2 = \mathrm{M+}\ 3 \div 2 + \mathrm{MRC} =$$

$$(1 \div 2) + (3 \div 2) = \boxed{2}$$

T: Do you know another way to write this calculation?

S: "$\frac{1}{2} + \frac{3}{2} = 2$".

Follow a similar procedure with the two problems below.

$$3 \div 2 + 1 \div 5 = \boxed{}$$

$$3 \div 2 = \mathrm{M+}\ 1 \div 5 + \mathrm{MRC} =$$

The class should conclude that the first calculation is for 0.5 while the second is for 1.7.

$$((3 \div 2) + 1) \div 5 = \boxed{0.5}$$

$$(3 \div 2) + (1 \div 5) = \frac{3}{2} + \frac{1}{5} = \boxed{1.7}$$

N 32

3.11 Addition of Rational Numbers #2

CAPSULE LESSON SUMMARY

Solve addition problems such as "$\frac{2}{5} + \frac{3}{4} = ?$" and "$\frac{5}{3} + \frac{3}{5} = ?$" first using appropriate partitions of regions. Then do similar calculations only thinking about the partitioning method, but finding appropriate (useful) fractional names for the numbers being added.

MATERIALS

Teacher: Colored chalk
Student : Colored pencils; Worksheets N47, N48, N49

DESCRIPTION OF LESSON

Exercise 1

Write this calculation on the board. Draw two regions the same size below the calculation.

$$\frac{2}{5} + \frac{3}{4} =$$

T: Let's do this addition problem with fractions and use Nabu's method of dividing the regions into rows and columns. How many rows?

S: Four* rows, because one of the numbers being added is $\frac{3}{4}$.

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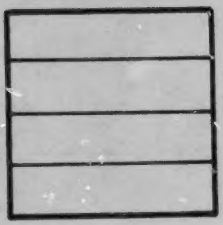
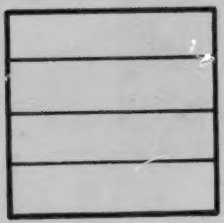
* Five rows and then four columns could also be used.

Ask a student to make the row partition of each region.

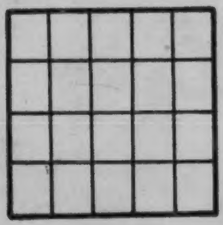


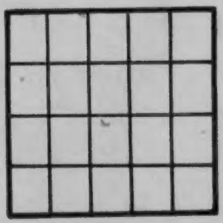


T: How many columns?

S: Five columns, because $\frac{2}{5}$ is the other number being added.

Ask a student to make the column partition of each region.





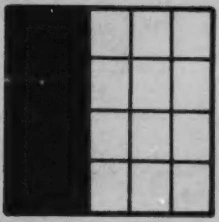
T: How many little pieces in each region?

S: 20; 4 × 5 = 20.

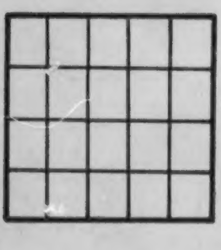
T (pointing to the region at the left): Shade two-fifths of this region.

Invite a student to do the shading and then to write "$\frac{2}{5}$" below the figure.

N 32



$\frac{2}{5}$



T: What fractional part of the whole region is each little piece?

S: One-twentieth.

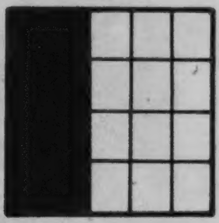
T: How many little pieces are shaded?

S: Eight.

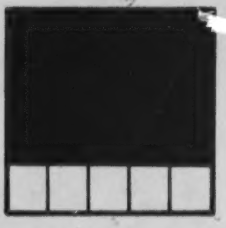
T: Does this picture suggest another name for $\frac{2}{5}$?

S: Yes, $\frac{2}{5} = \frac{8}{20}$.

Add this information to your drawing. In a similar manner, direct the class to shade three-fourths of the region at the right and to conclude that $\frac{3}{4} = \frac{15}{20}$.



$$\frac{2}{5} = \frac{8}{20}$$



$$\frac{3}{4} = \frac{15}{20}$$

T: Which is larger, $\frac{2}{5}$ or $\frac{3}{4}$? Why?

S: $\frac{3}{4}$ is larger; for $\frac{3}{4}$ fifteen pieces are shaded compared to eight pieces for $\frac{2}{5}$.

T: How many little pieces are shaded all together?

S: 23.

T: We said each little piece was one-twentieth of the region, so what fraction do we write for the shaded regions.

S: "$\frac{23}{20}$".

Complete the calculation on the board.

$$\frac{2}{5} + \frac{3}{4} = \frac{8}{20} + \frac{15}{20} = \frac{23}{20}$$

T: Is $\frac{23}{20}$ larger or smaller than 1?

S: Larger than 1; $\frac{20}{20} = 1$.

S: $\frac{23}{20} = 1\frac{3}{20}$.

Exercise 2

Write this calculation on the board.

$$\frac{1}{4} + \frac{5}{6} =$$

T: Nabu's method to do addition problems with fractions is to change the bottom numbers to the same number. Let's use Nabu's method, but only think about the drawings.

What are some other fractions for $\frac{1}{4}$?

N 32

Write the names for $\frac{1}{4}$ on the board as students suggest them.

$$\frac{1}{4} = \frac{2}{8} = \frac{4}{16} = \frac{3}{12} = \frac{5}{20} = \frac{8}{32} = \frac{6}{24} = \cdots$$

T: What are some other fractions for $\frac{5}{6}$?

Write these names on the board also.

$$\frac{5}{6} = \frac{10}{12} = \frac{20}{24} = \frac{15}{18} = \frac{30}{36} = \frac{40}{48} = \frac{35}{42} = \cdots$$

T: Which names for $\frac{1}{4}$ and $\frac{5}{6}$ are useful in doing the addition calculation?

Follow the students' suggestions to do the calculation one or more ways.

S: We could use "$\frac{3}{12}$" and "$\frac{10}{12}$".

$$\frac{1}{4} + \frac{5}{6} = \frac{3}{12} + \frac{10}{12} =$$

T: What number is $\frac{3}{12} + \frac{10}{12}$?

S: $\frac{13}{12}$.

$$\frac{1}{4} + \frac{5}{6} = \frac{3}{12} + \frac{10}{12} = \frac{13}{12}$$

S: We could use "$\frac{6}{24}$" and "$\frac{20}{24}$".

$$\frac{1}{4} + \frac{5}{6} = \frac{6}{24} + \frac{20}{24} =$$

T: What number is $\frac{6}{24} + \frac{20}{24}$?

S: $\frac{26}{24}$.

$$\frac{1}{4} + \frac{5}{6} = \frac{6}{24} + \frac{20}{24} = \frac{26}{24}$$

T: Can you think of another name for $\frac{26}{24}$?

S: "$1\frac{1}{12}$".

T: So there are many ways to calculate "$\frac{1}{4} + \frac{5}{6}$".

Exercise 3

Write this calculation on the board.

$$\frac{5}{3} + \frac{3}{5} =$$

Ask the students to copy this problem on their papers and to solve it by finding other names for $\frac{5}{3}$ and $\frac{3}{5}$. Check the problem collectively. The class should conclude:

$$\frac{5}{3} = \frac{10}{6} = \frac{15}{9} = \frac{35}{21} = \frac{25}{15} = \cdots$$

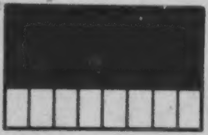
$$\frac{3}{5} = \frac{6}{10} = \frac{9}{15} = \frac{21}{35} = \frac{15}{25} = \cdots$$

$$\frac{5}{3} + \frac{3}{5} = \frac{25}{15} + \frac{9}{15} = \frac{34}{15} = 2\frac{4}{15}$$

Worksheets N47, N48, and N49 are available.

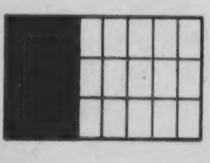
Name ____________________ N 47 *

Color two-thirds of this region red and fill in the box.



$$\frac{2}{3} = \frac{\boxed{16}}{24}$$

Color three-eighths of this region red and fill in the box.



$$\frac{3}{8} = \frac{\boxed{9}}{24}$$

Complete.

$$\frac{2}{3} + \frac{3}{8} = \frac{\boxed{16}}{24} + \frac{\boxed{9}}{24} = \underline{\frac{25}{24}}$$

(Other correct names for $\frac{25}{24}$ may be given; e.g., $\frac{25}{24} = 1\frac{1}{24}$.)

Name ____________________ N 48 **

Write at least five fractional names for $\frac{1}{5}$.

$$\frac{1}{5} = \frac{2}{10} = \frac{3}{15} = \frac{4}{20} = \frac{5}{25} = \frac{6}{30} = \cdots$$

Write at least five fractional names for $\frac{3}{2}$.

$$\frac{3}{2} = \frac{6}{4} = \frac{9}{6} = \frac{12}{8} = \frac{15}{10} = \frac{18}{12} = \cdots$$

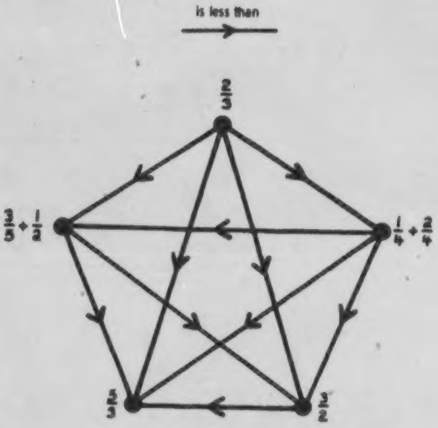
Complete.

$$\frac{1}{5} + \frac{3}{2} = \underline{\frac{2}{10} + \frac{15}{10} = \frac{17}{10}}$$

(Other correct names for $\frac{1}{5} + \frac{3}{2}$ could be given; e.g., $\frac{17}{10} = 1\frac{7}{10}$.)

Name ____________________ N 49 ***

Draw as many red arrows as possible.



4

ABACUSES AND ALGORITHMIC THINKING

At the core of traditional elementary school mathematics is the development of algorithms for the basic operations: addition, subtraction multiplication, and division. The idea of algorithm itself is important in mathematics; however, the complexity or usualness of the standard algorithms make them unsuitable for students to appreciate the nature of an algorithm, to relate them to other algorithms, or to develop algorithmic thinking. In this chapter, the lessons explore different abacuses for developing specialized algorithms for division by 9 and for binary calculations.

CAPSULE LESSON SUMMARIES

Section 4.1:	Introduce a new abacus and with it develop an algorithm for dividing by 9. Examine the abacus by putting numbers on it, decoding numbers, and making plays (according to its rules) with both regular and weighted checkers. Solve some problems involving division by 9 with the help of the abacus.
Section 4.2:	Review the rule of the binary abacus. Practice putting and reading numbers on it and making plays. Observe that on the abacus doubling is an easy calculation. Introduce binary notation as "Nabu's Code" for numbers and do some addition calculations in the binary system.
Section 4.3:	Review the rules of the new abacus (introduced in Section 4.1) and practice putting numbers on it, decoding numbers, and making plays. Examine special configurations for numbers which allow the abacus to provide an algorithm for dividing by 9. Solve some problems involving division by 9 on the abacus.

Section 4.4: Review the rule of the binary abacus and Nabu's (binary) code for numbers. Practice writing numbers in the binary code and decoding numbers written in binary notation. Write addition and multiplication calculations in the binary system. Discover the binary code for $\frac{1}{3}$ and $\frac{1}{5}$.

4.1 Nabu's Abacus #1

CAPSULE LESSON SUMMARY

Introduce a new abacus and with it develop an algorithm for dividing by 9. Examine the abacus by putting numbers on it, decoding numbers, and making plays (according to its rules) with both regular and weighted checkers. Solve some problems involving division by 9 with the help of the abacus.

MATERIALS

Teacher: Colored chalk; demonstration Minicomputer checkers (optional); weighted checkers

Student: None

DESCRIPTION OF LESSON

Begin the lesson with a short warm-up activity consisting primarily of mental arithmetic for multiplication by 9. Include these and other calculations. appropriate to the numerical level of your class.

9 x 4;	9 x 7;	9 x 13;	9 x 22;
9 x 5;	9 x 8;	9 x 14;	9 x 23;
9 x 6;	9 x 9;	9 x 15;	9 x 24.

Exercise 1

T: I want to tell you about Nabu and his summer job at a packing plant. Nabu is given a number of crayons and his job is to put the crayons into nine boxes in such a way that each box has the same number of crayons and each box has as many crayons as possible. His first assignment is to put 40 crayons into nine boxes. How many crayons should he put in each box and how many crayons are left over?

Let the students have a minute to consider the problem.

S: He should put four crayons in each box and have four crayons left over; (9 x 4) + 4 = 36.

T: Next, Nabu is given 57 crayons to put into nine boxes. How many crayons does he put in each box and how many crayons are left over?

S: Six in each box with three left over, because 9 × 6 = 54 and 54 + 3 = 57.

Continue in this manner using the examples suggested below or others better suited to the numerical level of your class.

$$75 = (9 \times 8) + 3$$
$$130 = (9 \times 14) + 4$$
$$212 = (9 \times 23) + 5$$

As the calculations become more difficult to perform mentally, encourage the students to use estimation. For example,

T: Nabu is given 212 crayons to put into nine boxes. How many crayons should he put in each box and how many are left over?

Could he put 10 crayons into each box?

S: He can put more than 10 crayons in a box, because 9 × 10 = 90.

T: Could he put 20 crayons into each box?

S: He can put more than 20 into each box, because 9 × 20 = 180.

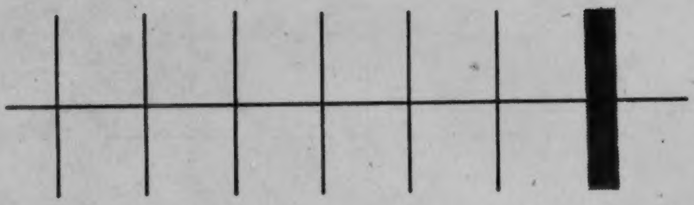
T: Could he put 30 crayons into each box?

S: No. 9 × 30 = 270.

S: He should put 23 into each box and have 5 left over, because (9 × 23) + 5 = 212.

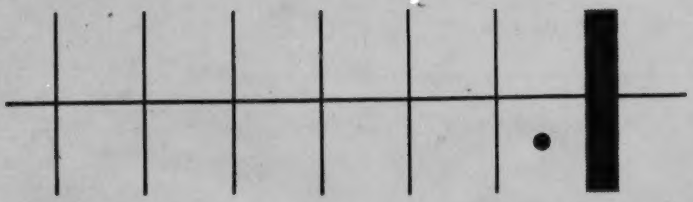
Exercise 2

Draw this abacus on the board.



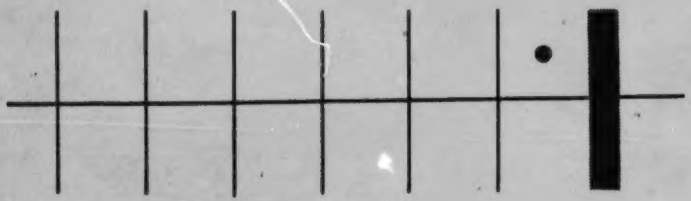
T: Nabu wanted to find a simple way to solve his division problems. So he invented a new abacus, which I have drawn on the board.

Place a checker on the 1-square as shown in the illustration below. If your board is not magnetic, draw the checkers.



T: This is the number 1.

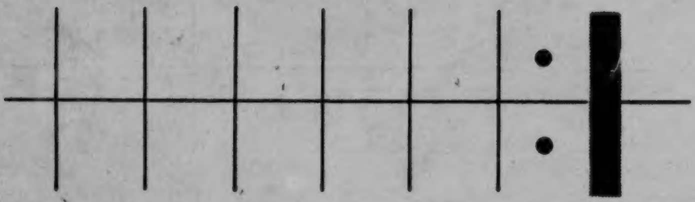
Move the checker to the square directly above.



N 5

T: This is 9.

Add a checker to the 1-square.

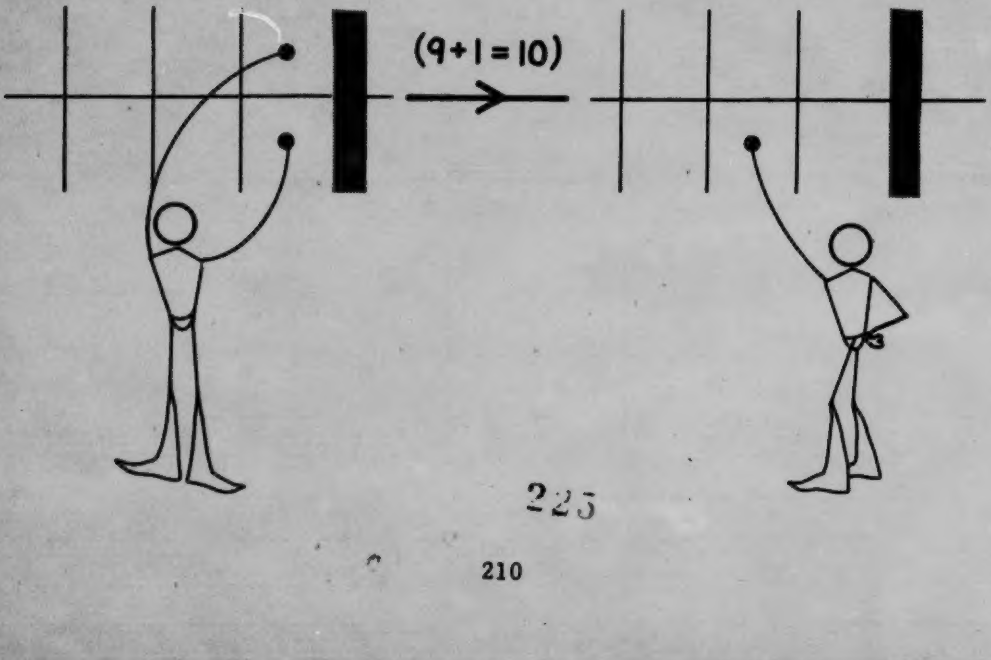


T: What number is this?

S: 10.

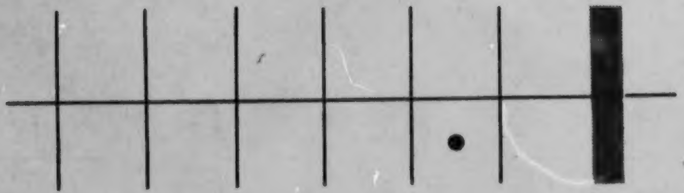
T: There is another way to put 10 on Nabu's abacus with just one checker. Do you know how to do it?

Invite a student to make the play illustrated below, or demonstrate it yourself.

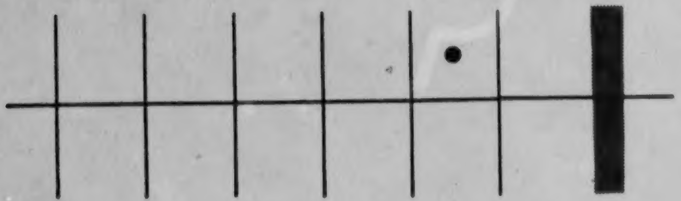


225

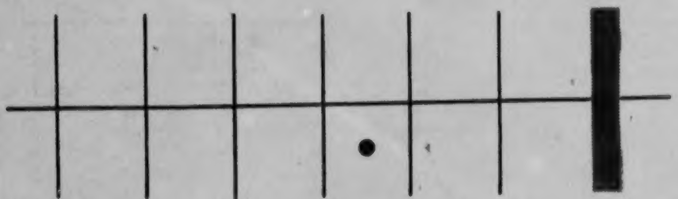
Continue the activity with these configurations.



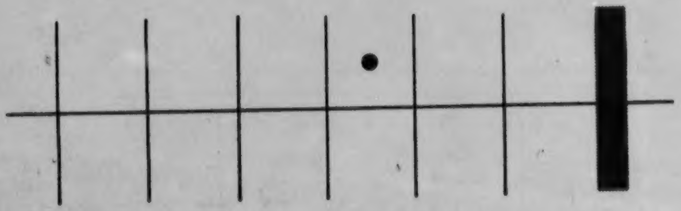
= 10



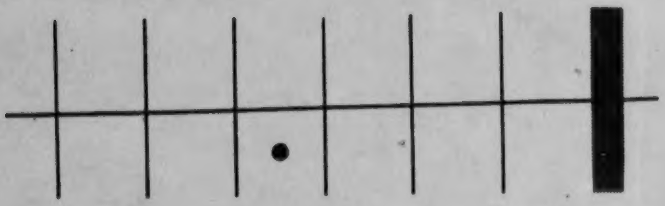
= 90



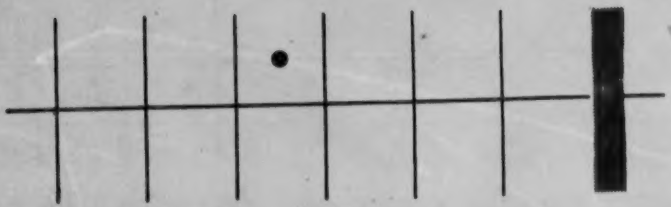
= 100



= 900



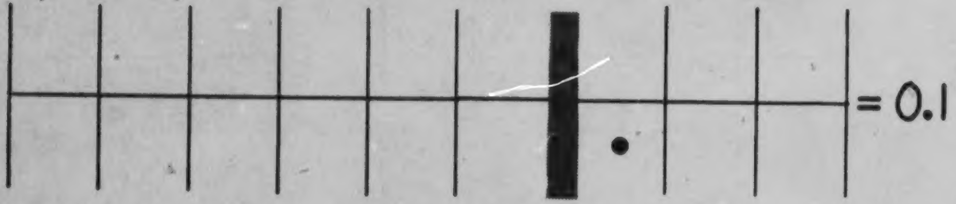
= 1,000

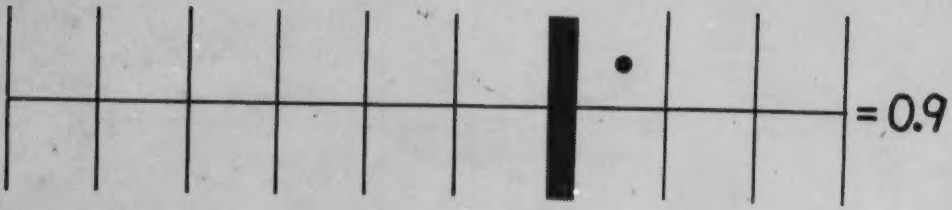


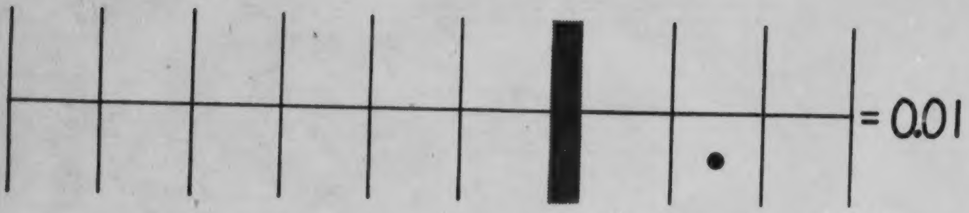
= 9,000

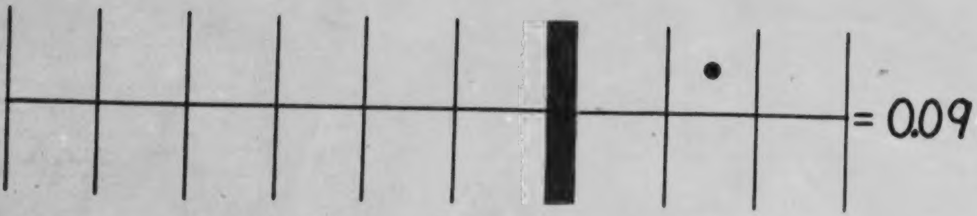
N 5

When it is clear that most students have discovered the pattern of Nabu's abacus, extend the abacus to the right of the bar. Then ask the class to decode these numbers.

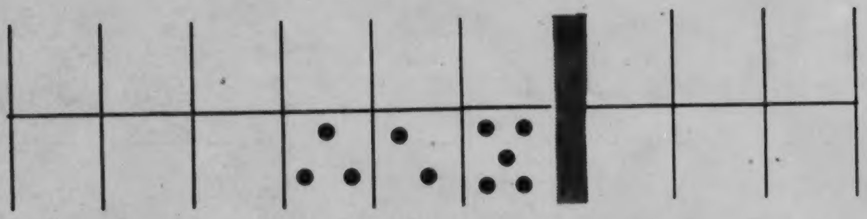






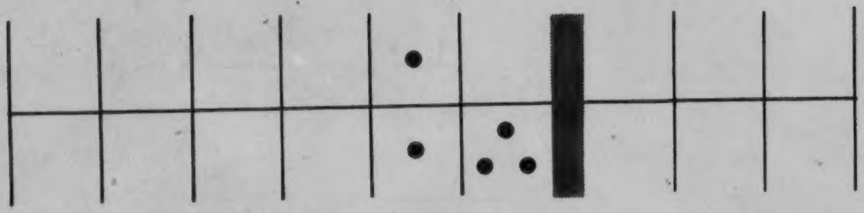


Display this configuration of checkers.



T: What number is this?

S: 325.

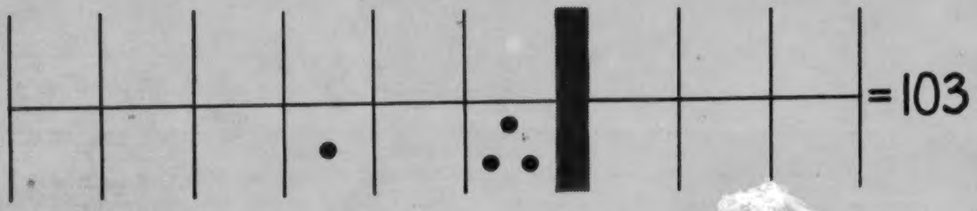


T: What number is this?

S: 103, because 90 + 10 + 3 = 103.

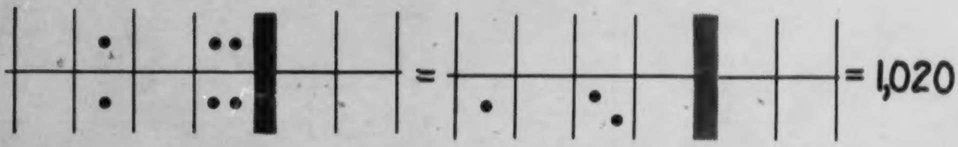
T: Good. Who can show me the same number but with one checker less?

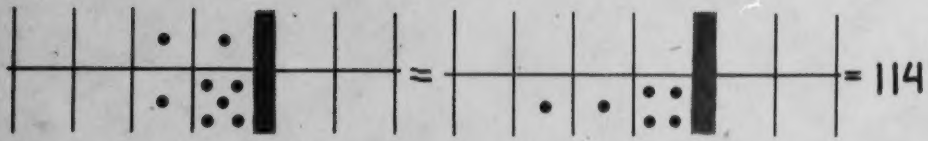
Invite a student to make a play ("90 + 10 = 100").

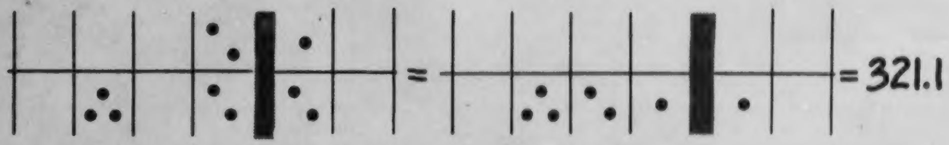


N 5

Continue this activity using the configurations provided below. For each situation ask students to make plays and to announce which plays they make.



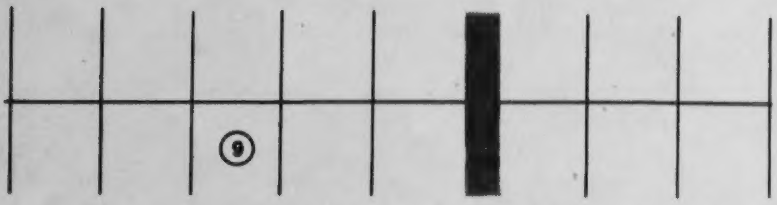




Exercise 3

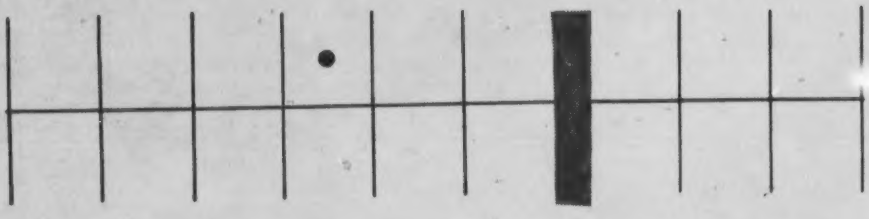
For this exercise, you will need a set of weighted checkers (② - ⑩) in addition to a set of regular checkers.

Display this configuration on the abacus.

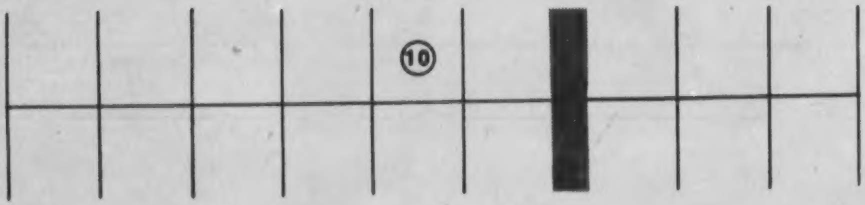


T: Who can show us the same number using a regular checker rather than this ⑨ -checker?

Call on a student to make the trade.



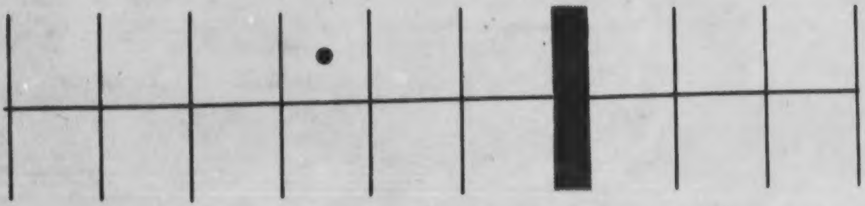
T: Who can trade this regular checker for a ⑩ -checker?



T: Write this number on your paper.

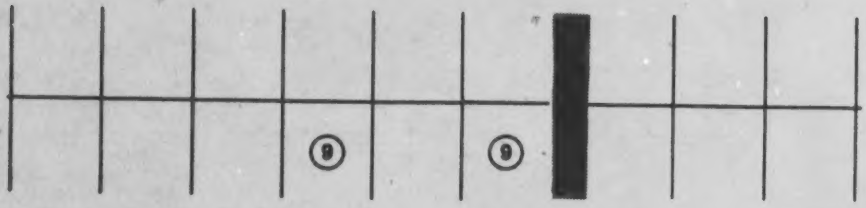
S: 900, because 10 × 90 = 900.

T: Who can trade this ⑩-checker for a regular checker?



N 5

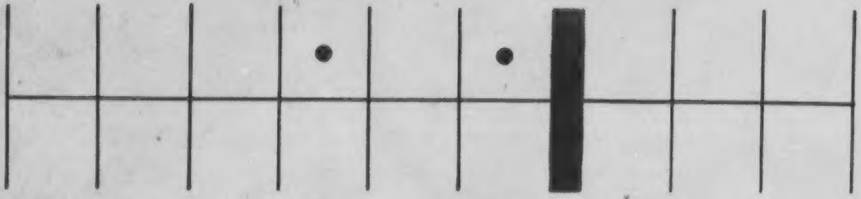
Clear the abacus and display this configuration.



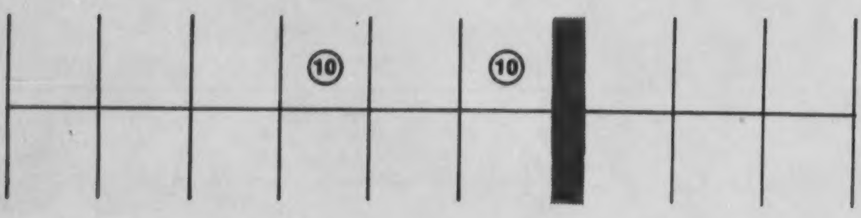
T: What number is this?

S: 909.

T: Who can trade the two ⑨ - checkers for two regular checkers?



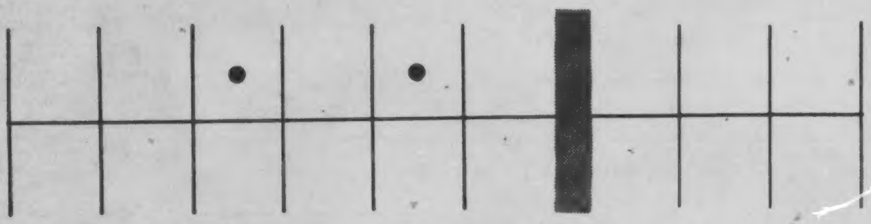
Clear the abacus and display this configuration.



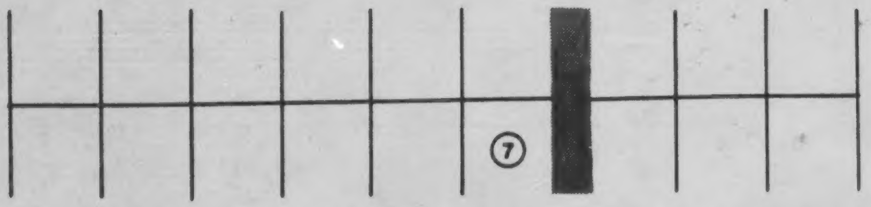
T: Write this number on your paper.

S: 9,090; because 10 x 909 = 9,090.

T: Who can trade the two ⑩-checkers for two regular checkers?



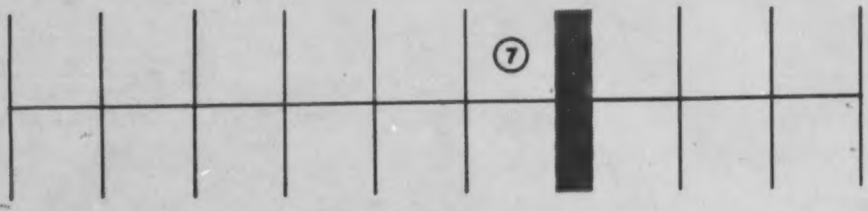
Clear the abacus and put on this configuration.



T: What number is this?

S: 7.

Move the ⑦-checker to the square above.



T: What number is this?

S: 63.

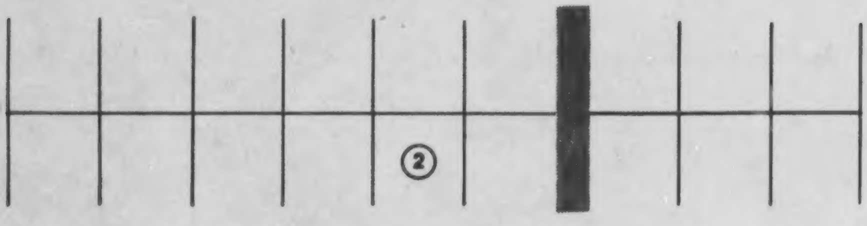
N 5

T: How many times 7 is 63?

S: 9 × 7 = 63.

Write this number sentence on the board and clear the abacus.

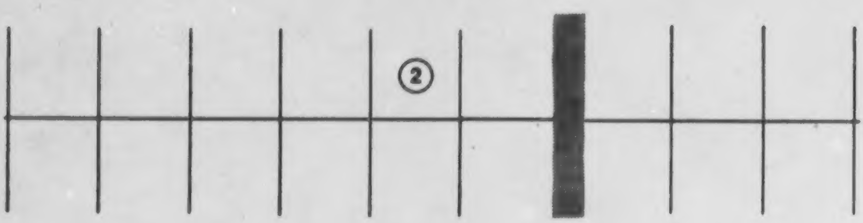
Put a ② -checker on the 10-square.



T: What number is this?

S: 20, because 2 × 10 = 20.

Move the ② -checker to the square above.



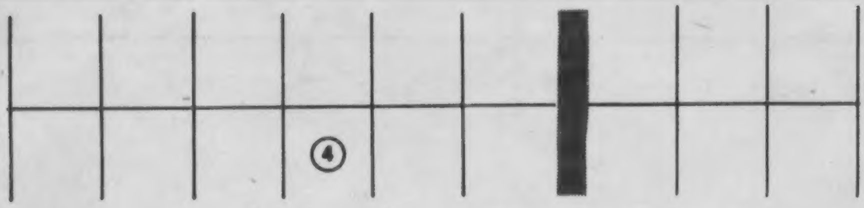
T: What number is this?

S: 180.

T: How many times 20 is 180?

S: 9 × 20 = 180.

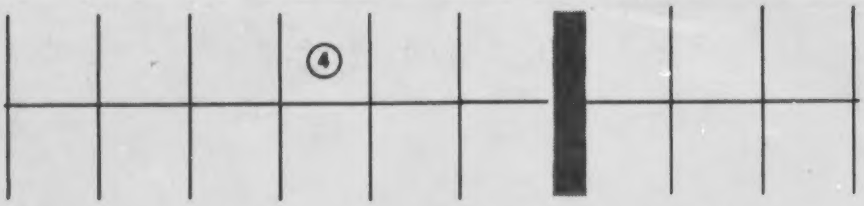
Write this number sentence on the board and clear the abacus. Continue this activity by putting a ④-checker on the 100-square.



T: What number is this?

S: 400.

Move the ④-checker to the square above.



T: And what number is this?

S: 3,600.

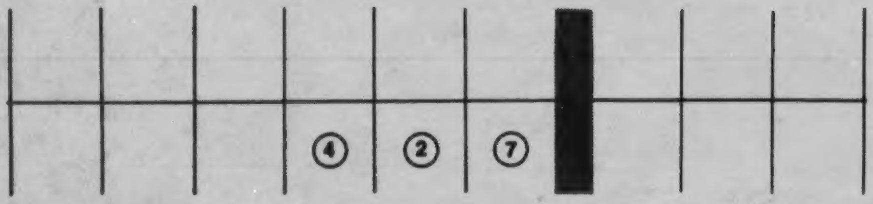
T: How many times 400 is 3,600?

S: 9 times, because 9 × 400 = 3,600.

N 5

Write the number sentence on the board.

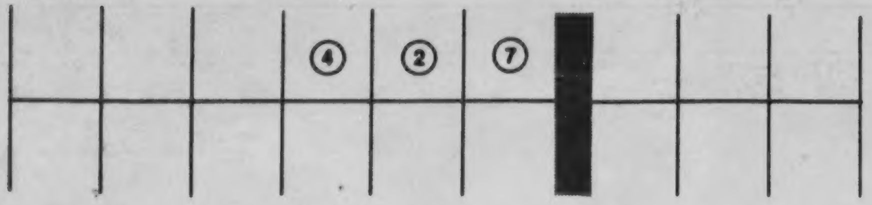
Put this configuration on the abacus.



T: What number is this?

S: 427.

T: Now I will move each checker to the square above. What can you tell me about the new number?



Give the class a few minutes to consider this situation before asking for a response.

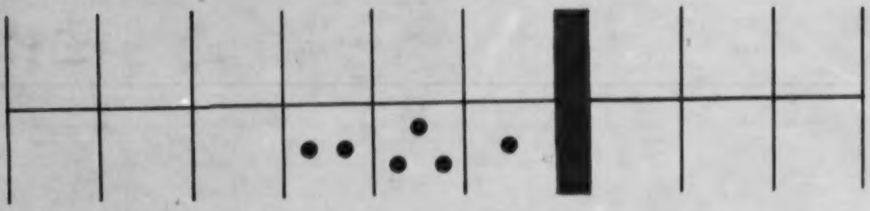
S: The number is 9 x 427.

S: 3,600 + 180 + 63 = 3,843.

S: 9 x 427 = 3,843.

If there is still time available, try this optional activity.

Put this configuration on the abacus.

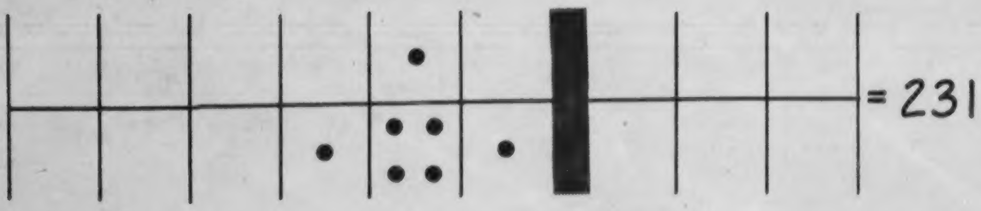


T: What number is on the abacus?

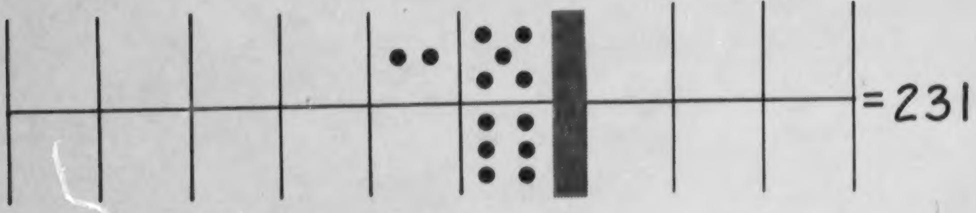
S: 231.

T: Good. Who can show the same number using one checker more?

Call on a student to make a backward play. For example, "100 = 90 + 10".

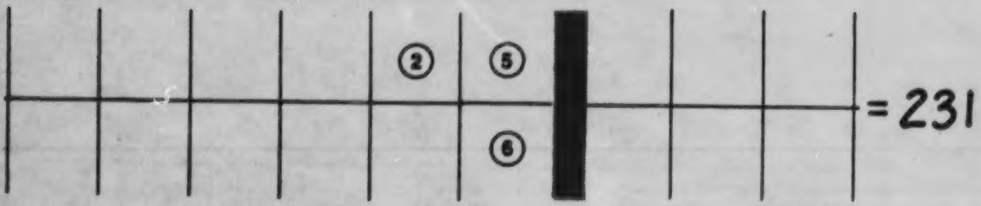


Continue asking students to show the same number using one checker more until this configuration is obtained.



N 5

Replace the regular checkers with the appropriate weighted checkers.



T: **Nabu has 231 crayons to put into nine boxes. He claims that by looking at the abacus with 231 on it in this special way, he knows how many crayons to put into each box. Do you see the solution to Nabu's problem?**

Let the students have several minutes to consider this situation and express their opinions.

It is likely that many students will solve Nabu's problem without the help of the abacus (25 crayons in each box with 6 leftover). Encourage them to look for some relationship between their answer and the checkers on the abacus.

Some students might observe that the abacus shows 231 = (9 x 25) + 6.

If no explanation of Nabu's claim is made, it is not necessary for you to give one. In a later lesson, this situation will again be explored.

4.2 Binary Code #1

CAPSULE LESSON SUMMARY

Review the rule of the binary abacus. Practice putting and reading numbers on it and making plays. Observe that on the abacus doubling is an easy calculation. Introduce binary notation as "Nabu's Code" for numbers and do some addition calculations in the binary system.

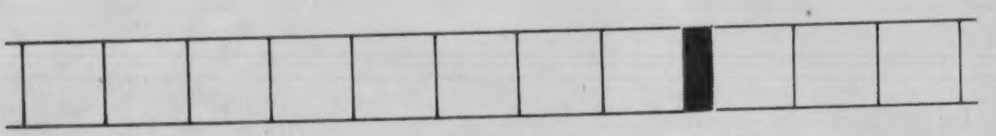
MATERIALS

Teacher: Colored chalk; demonstration Minicomputer checkers
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Draw this part of an abacus on the chalkboard or use Minicomputer boards.*



T: Do you remember the binary abacus and its rule?

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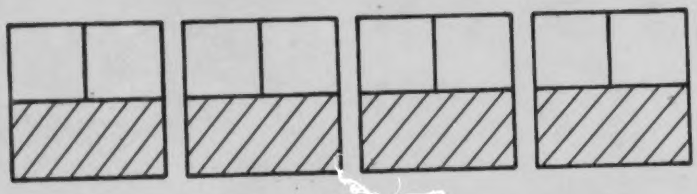
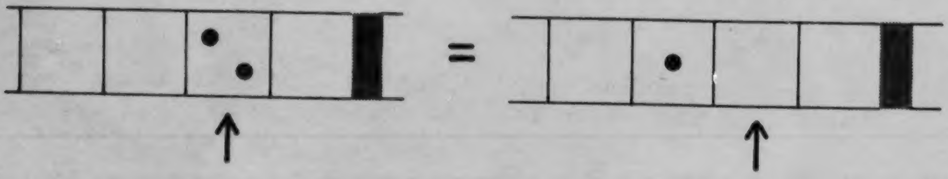
* If your chalkboard is not magnetic, a magnetic abacus can be constructed using your demonstration Minicomputer boards. Cover the red and white squares with paper and use only the brown and purple squares on each board.



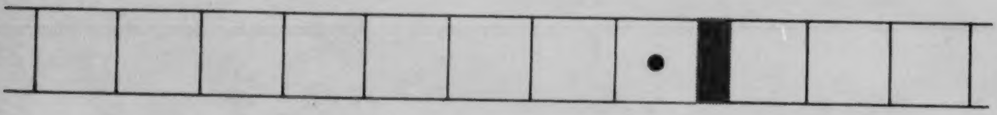
N 9

Invite a student to explain the binary abacus rule to the class. The class should be reminded that the rule is :



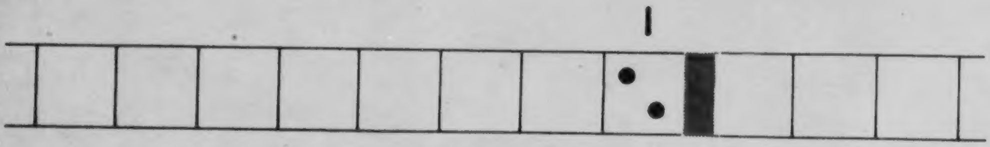
Two checkers on a board represents the same number as one checker on the next board to the left.

Put (draw) one checker on the abacus as shown here.



T:	This is the number 1.

Label the board "1" and put another checker on the ones' board.

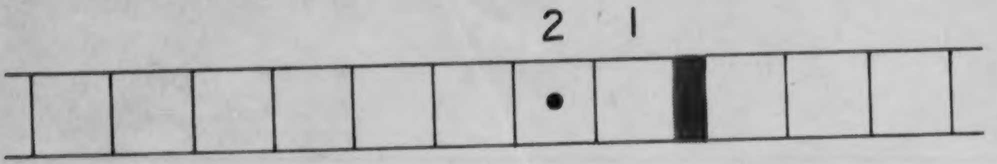


T:	What number is this?

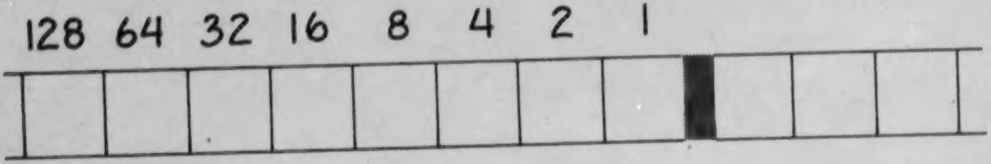
S:	2.

T:	Can you put 2 on the abacus using exactly one checker?

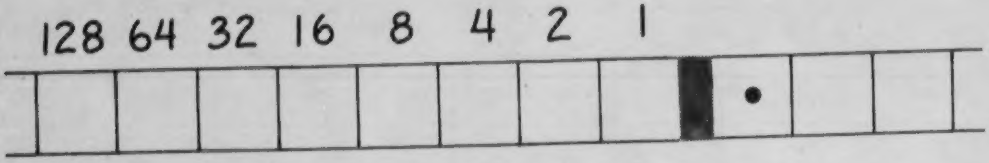
Ask a student to put 2 on this abacus with exactly one checker and then label the board "2".



Continue until you have labeled each board to the left of the bar.

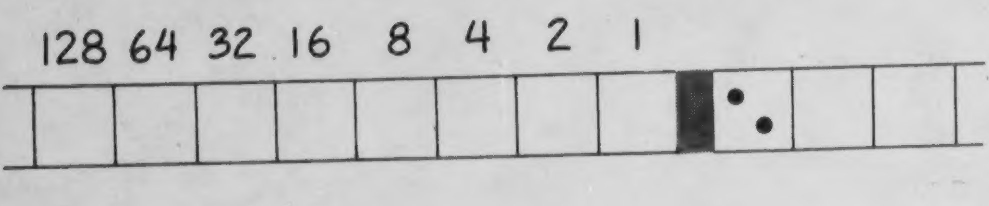

128
64
32
16
8
4
2
1

Put one checker on the abacus as shown here.


128
64
32
16
8
4
2
1

T: What number is this?

To remind students who have difficulty with this situation, put another checker on the same board.


128
64
32
16
8
4
2
1

T: What number is this?

N 9

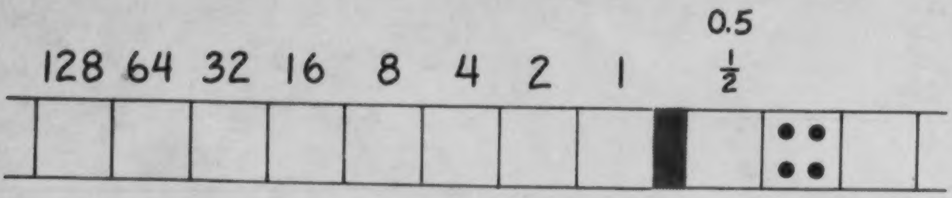
If necessary, ask a student make a play to show that the number is 1.

S: 1.

T: Two checkers on this board is the number 1, so one checker on this board is which number?

S: 0.5 or $\frac{1}{2}$.

Label the board both "0.5" and "$\frac{1}{2}$". Then display this configuration of checkers.



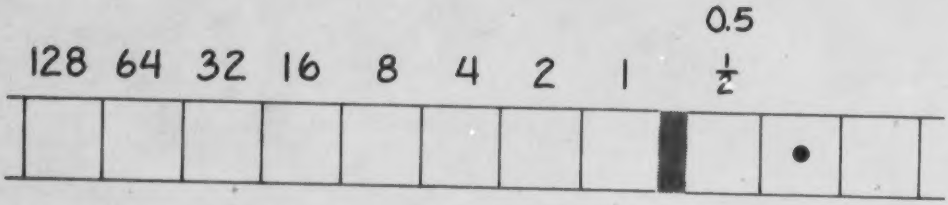
T: What number is this?

S: 1.

If necessary, invite a student to make the plays to show that the number is 1.

T: Suppose I put one checker on this same board.

Put one checker on the abacus as shown below.



T: What number is this?

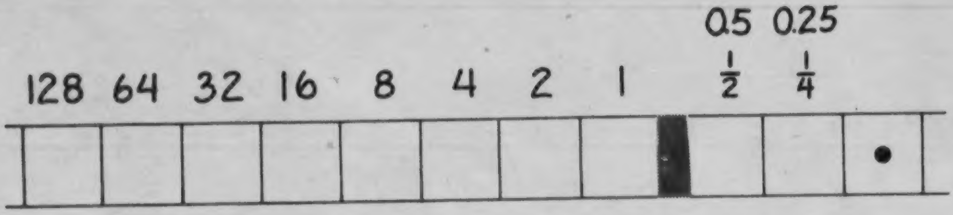
S: $\frac{1}{4}$ or 0.25.

Label the board both "$\frac{1}{4}$" and "0.25".

T (pointing to the third board to the right of the bar): How many checkers does it take on this board to make the number 1?

S: Eight.

Put one checker on the board.



T: Eight checkers on this board is the number 1. What is this number?

S: $\frac{1}{8}$.

T: Who knows a decimal name for $\frac{1}{8}$?

You might give the class a hint by rewriting "0.5" and "0.25" as "0.500" and "0.250", respectively.

S: 0.125.

Remove the checkers from the abacus. Ask the students to draw an abacus on their papers and to put 45 on the abacus using at most one checker on any board.

N 9

After a minute or two, invite a student who is finished to put 45 on the demonstration abacus using at most one checker on any board.

128	64	32	16	8	4	2	1	0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$
		●		●	●		●			

= 45

T: Now quickly, who can put 90 on the abacus?

S: Move each checker one board to the left.

128	64	32	16	8	4	2	1	0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$
	●		●	●		●				

= 90

T: Now 180, very quickly.

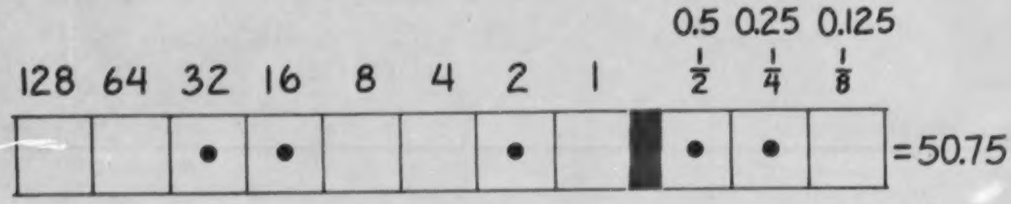
S: Move each checker one board to the left.

128	64	32	16	8	4	2	1	0.5 $\frac{1}{2}$	0.25 $\frac{1}{4}$	0.125 $\frac{1}{8}$
●		●	●		●					

= 180

T: Draw another abacus on your paper and put 50.75 on it. Remember, use at most one checker on a board.

After a few minutes of working independently, invite a student who is finished to put 50.75 on the demonstration abacus using at most one checker on a board.



T: I will move each checker one board to the left. How can you very easily calculate the new number?

S: Multiply by 2, because moving the checkers one board to the left doubles the number.

T: What is 2 × 50.75?

S: 2 × 50 = 100 and 2 × 0.75 = 1.5, so 2 × 50.75 = 101.5.

If you think it necessary, write this problem on the board and solve it in the usual manner.

Exercise 2

NOTE: You will notice that whenever we write numbers in the binary code, we use "1" and whenever we write numbers in the decimal code, we use "|". This device serves as a visual reminder to the students that we are writing in a code different from the decimal code; there is no mathematical reason for it.

Write this information on the board.

NABU'S CODE		DECIMAL WRITING
1101	=	13
111	=	7
1001	=	☐

N 9

T: Nabu has invented a strange code to write numbers. I will not explain it to you. You will have to try to crack his code.

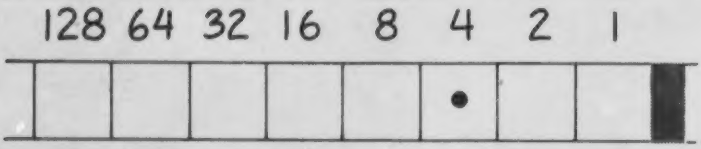
What number is this (point to the box)?

Walk around the room and let students whisper their guesses in your ear. Acknowledge aloud any correct guess and deny any incorrect one. For example, "Yes, that's correct." or "No, 5 is not the number." If no one suggests 9, do so yourself. Continue the activity with the problems below (answers are in the boxes). Tell students who need a hint that Nabu's code has something to do with the binary abacus.

NABU'S CODE		DECIMAL WRITING
1111	=	$\boxed{15}$
110	=	$\boxed{6}$
101	=	$\boxed{5}$
11	=	$\boxed{3}$
$\boxed{100}$	=	4

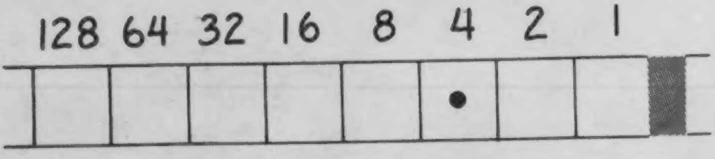
For the benefit of students who have not yet cracked Nabu's code, do the following activity.

T: Who can put 4 on the binary abacus?



T: Now, who can write Nabu's code name for 4?

Ask a student to write the answer below the abacus.



NABU'S CODE		DECIMAL WRITING
1 0 0	=	4

Continue the activity with these problems. If necessary, use the binary abacus as an aid.

NABU'S CODE		DECIMAL WRITING
100001	=	$\boxed{33}$
$\boxed{1101}$	=	13
110011	=	$\boxed{51}$
$\boxed{11001}$	=	25
101.1	=	$\boxed{5.5}$

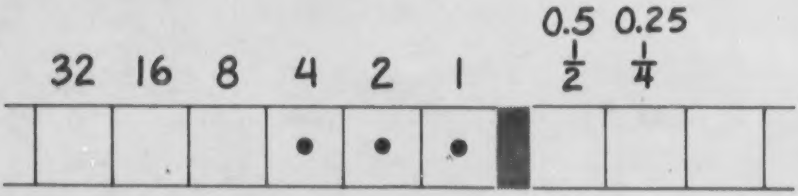
If you would like to challenge your class, ask them to solve this set of problems.

NABU'S CODE		DECIMAL WRITING
110.01	=	$\boxed{6.25}$
$\boxed{111.11}$	=	7.75
110.001	=	$\boxed{6.125}$
1001.101	=	$\boxed{9.625}$

N 9

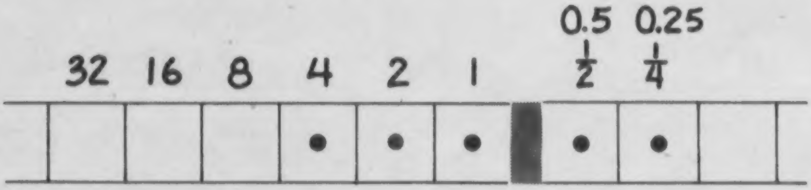
Use the binary abacus to check answers. For example,

T: We must put 7.75 on the abacus. First let's put 7 on the abacus.

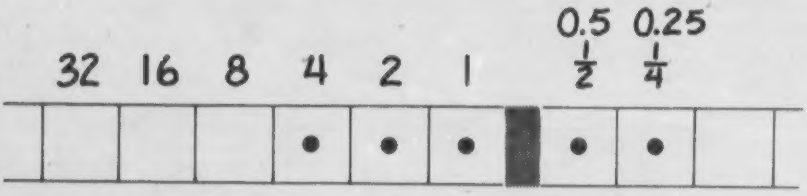

0.5
0.25
32
16
8
4
2
1
$\frac{1}{2}$
$\frac{1}{4}$

T: How can we put 0.75 on the abacus?

S.: Use "0.75 = 0.50 + 0.25".


0.5
0.25
32
16
8
4
2
1
$\frac{1}{2}$
$\frac{1}{4}$

T: Now, what is Nabu's code for 7.75?


0.5
0.25
32
16
8
4
2
1
$\frac{1}{2}$
$\frac{1}{4}$

NABU'S CODE DECIMAL WRITING

1 1 1 . 1 1 = 7.75

Exercise 3

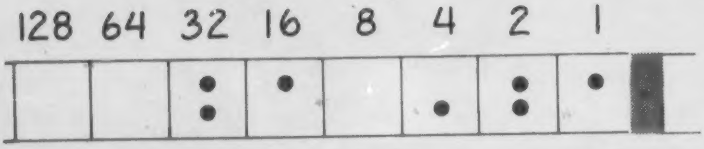
Write this addition problem on the board.

$$\begin{array}{r} 110011 \\ +\ 100110 \\ \hline \end{array}$$

T: I have written an addition problem in the binary code. How do you suggest we solve this problem?

Some students may be able to solve the problem directly, while others will need the support of the binary abacus.

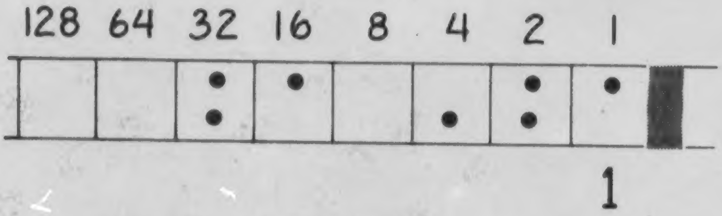
Put the addends on the abacus, using a different color checker for each.



T: Look carefully at each board of the abacus. There is only one checker on the ones' board so no play need be made with that checker.

N 9

As you point this out to the students, write this information in the appropriate places under the abacus and in the addition calculation.

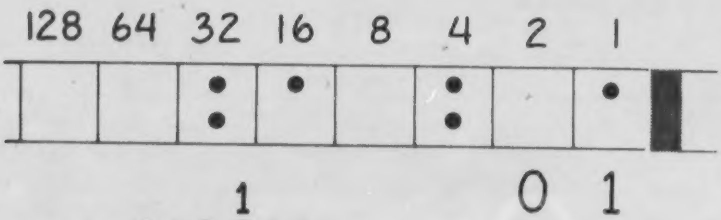


$$\begin{array}{rcccccc} & 1 & 1 & 0 & 0 & 1 & 1 \\ + & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline & & & & & & 1 \end{array}$$

T: Now look at the twos' board. There are two checkers on this board so we must make a move; this is the same as adding 1 and 1.

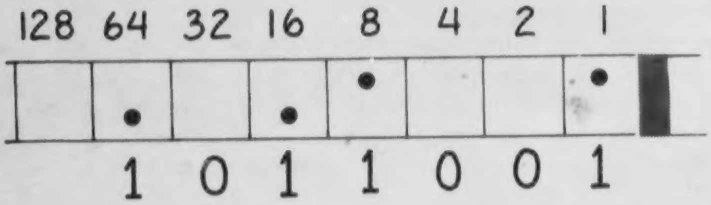
Make the play and add this information to the appropriate places.

NOTE: After the initial placement of checkers on the abacus, their colors are no longer important. For that reason, the pictures from here on use black checkers.



$$\begin{array}{rcccccc} & & & & 1 & & \\ & 1 & 1 & 0 & 0 & 1 & 1 \\ + & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline & & & & & 0 & 1 \end{array}$$

Continue working from the abacus to the calculation until you have reached this configuration and have completed the calculation.



$$\begin{array}{r} 11\overset{1}{0}\overset{1}{0}11 \\ +\ 100110 \\ \hline 1011001 \end{array}$$

Continue the activity with these problems. Allow students a chance to do the calculations on their papers and then check the solutions collectively.

$$\begin{array}{r} 111011 \\ +\ 111110 \\ \hline \boxed{1111001} \end{array} \qquad \begin{array}{r} 101.011 \\ +\ 111.101 \\ \hline \boxed{1101.000} \end{array}$$

N 16

4.3 Nabu's Abacus #2

CAPSULE LESSON SUMMARY

Review the rules of the new abacus (introduced in Lesson 4.1) and practice putting numbers on it, decoding numbers, and making plays. Examine special configurations for numbers which allow the abacus to provide an algorithm for dividing by 9. Solve some problems involving division by 9 on the abacus.

MATERIALS

Teacher:	Colored chalk; demonstration Minicomputer checkers (optional); weighted checkers
Student:	Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

T: Do you remember Nabu and his summer job?

Let the class recall that Nabu is working in a packing plant. Nabu is given a number of crayons and his job is to put the crayons into nine boxes in such a way that each box has the same number of crayons and each box has as many crayons as possible.

T: Nabu must put 52 crayons into nine boxes. How many crayons should he put in each box and how many crayons are left over?

Let the students have a minute to consider the problem.

S: He puts five crayons in each box and seven crayons are left over; $(9 \times 5) + 7 = 52$.

Continue in this manner using the examples suggested below or other examples better suited to the numerical level of your class.

$$73 = (9 \times 8) + 1$$

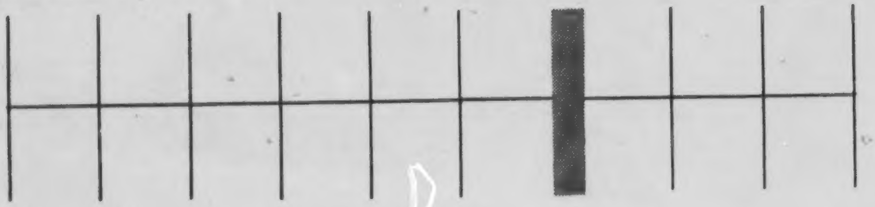
$$110 = (9 \times 12) + 2$$

$$210 = (9 \times 23) + 3$$

Exercise 2

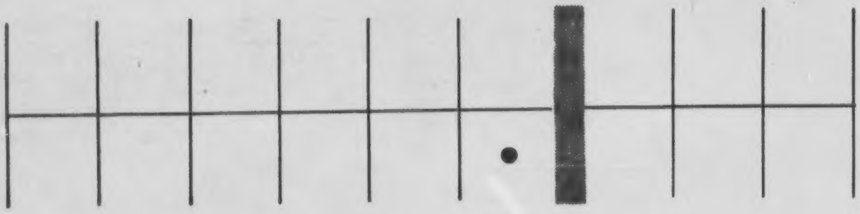
For this exercise, you will need a set of weighted checkers (② - ⑩) in addition to a set of regular checkers.

Draw this abacus on the board.



T: Do you remember the abacus Nabu invented to help him with the division problems?

Put (draw) a checker on the 1-square as shown in the next illustration.

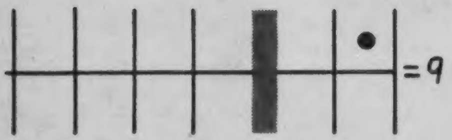


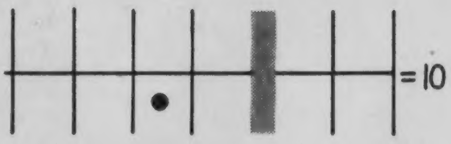
N 16

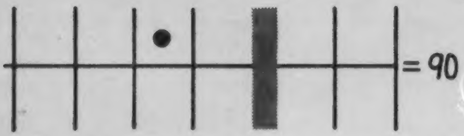
T: Which number is this?

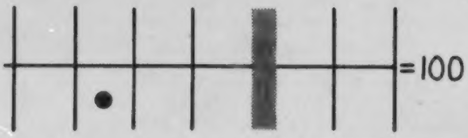
S: 1.

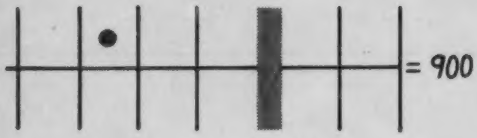
Continue reviewing the values of the squares by moving the checker and asking for the number.

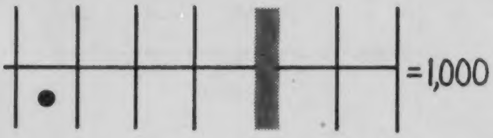


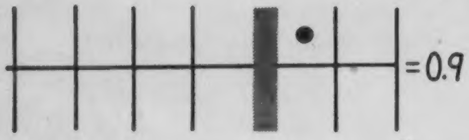


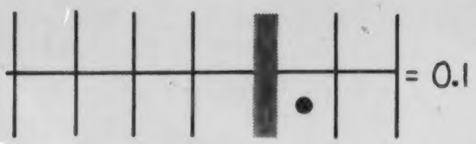


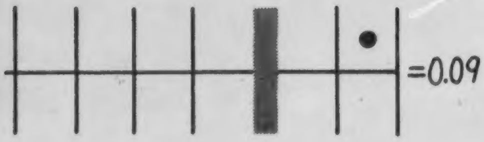


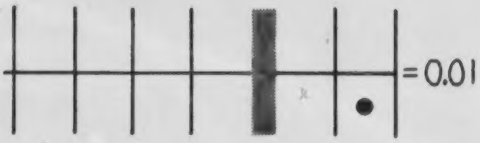




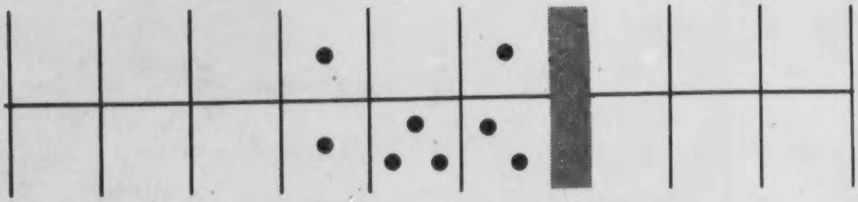








Display this configuration of checkers.

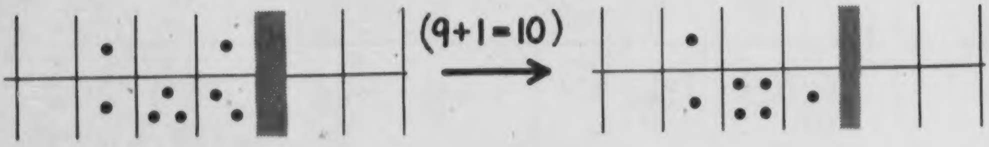


T: Write this number on your paper.

Let the students have a few minutes to work independently, and then ask:

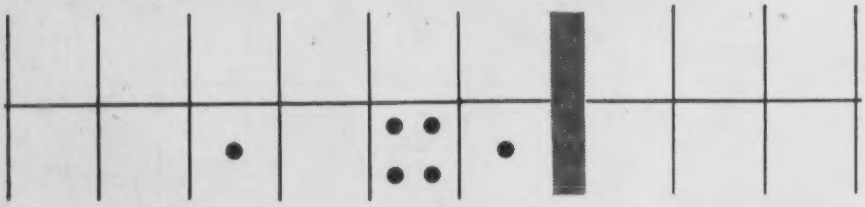
T: Who can show the same number but with one checker less?

Invite a student to make a play; for example,



T: Good. Show the same number again with one checker less.

S: 900 + 100 = 1,000.

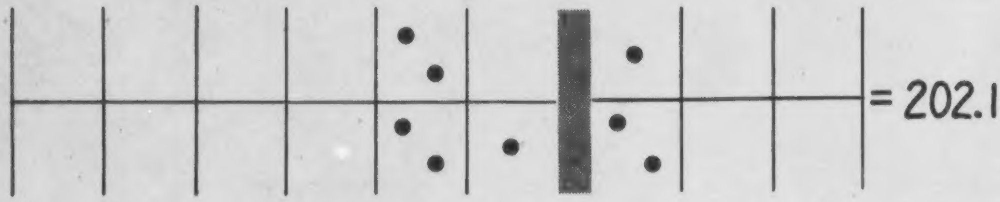


N 16

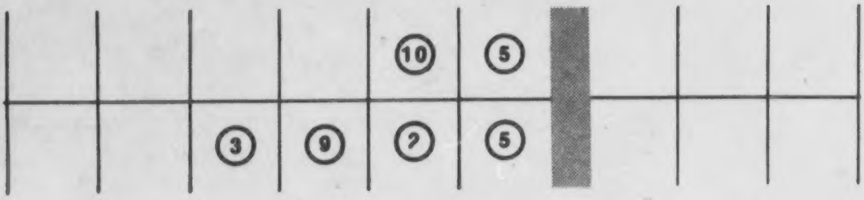
T: It is easy to read the number on the abacus now. What number is it?

S: 1,041.

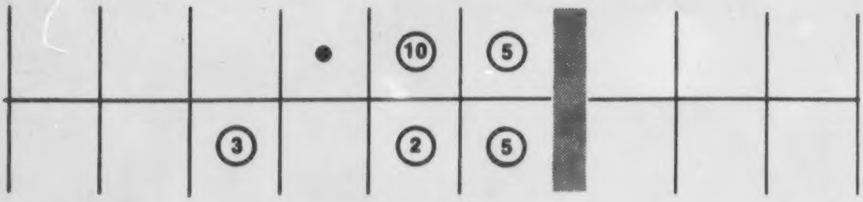
Repeat this activity starting with the following configuration.



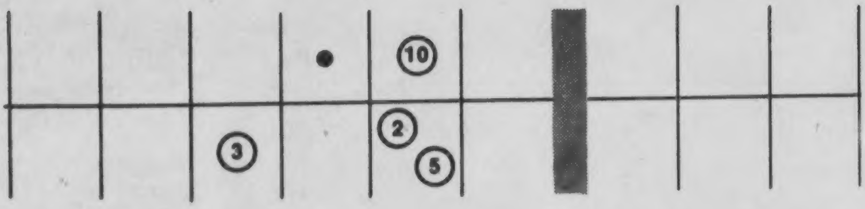
Display this configuration on the abacus.



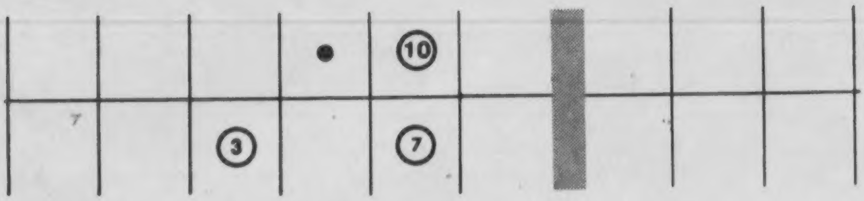
T: Who can trade the ⑨-checker for a regular checker and show the same number ?



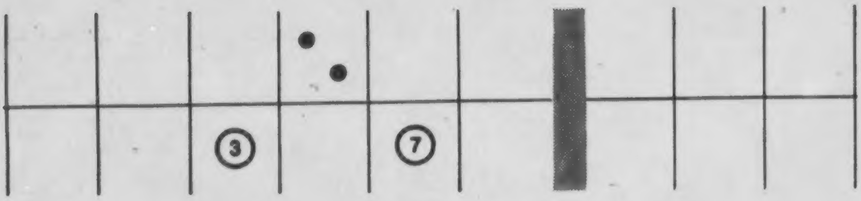
T: Who can trade two ⑤-checkers for one ⑤-checker and show the same number?



T: Can someone trade the ⑤-checker and the ②-checker for one other checker?

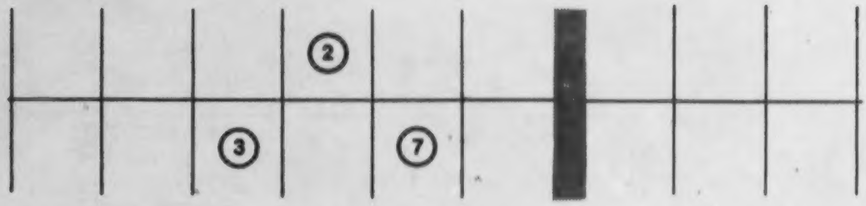


T: Who can trade the ⑩-checker for a regular checker and show the same number?



N 16

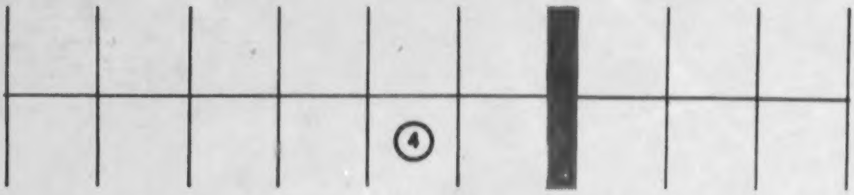
T: Who can trade these two regular checkers for one other checker and show the same number?



T: Which number is on the abacus?

S: 4,870; because 3,000 + 1,800 + 70 = 4,870.

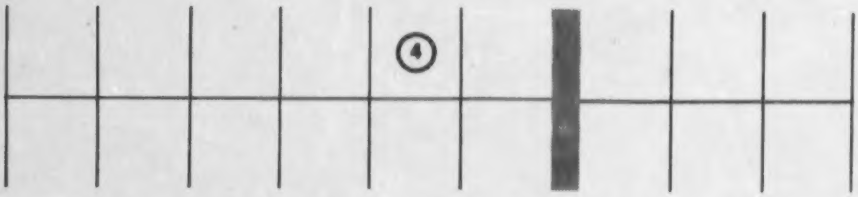
Clear the abacus and put a ④-checker on the 10-square.



T: Which number is this?

S: 40.

T: Now, I move the checker to the square above. Which number is this?



257

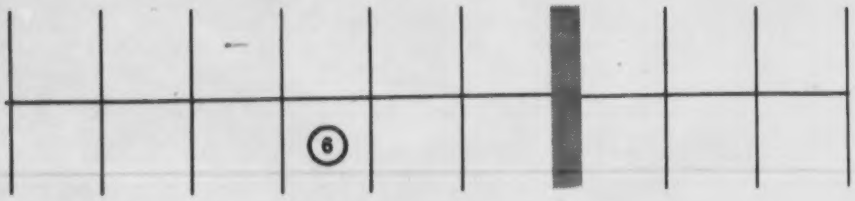
S: 360; because 4 × 90 = 360.

T: How many times 40 is 360?

S: 9; 9 × 40 = 360.

Write "9 × 40 = 360" on the board.

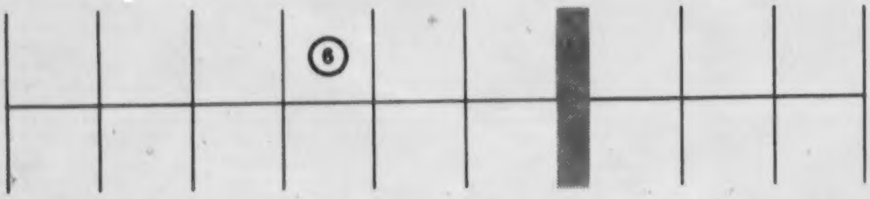
Continue this line of questioning starting with a ⑥-checker on the 100-square.



T: Which number is this?

S: 600.

Move the ⑥-checker to the square above.



T: Which number is this?

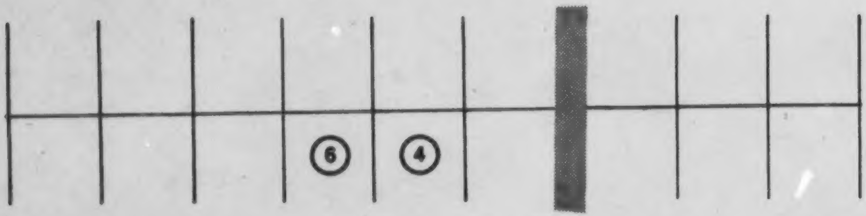
S: 5,400; because 6 × 900 = 5,400.

N 16

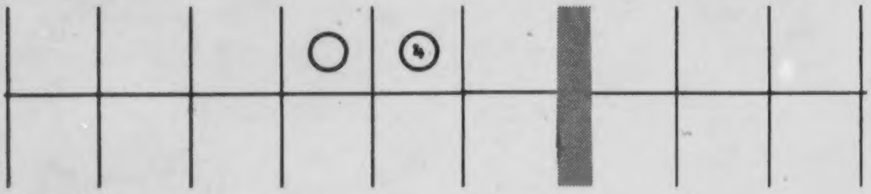
T: How many times 600 is 5,400?

S: 9; 9 × 600 = 5,400.

Put this configuration on the abacus.



T: I put 640 on the abacus. Now I will move each checker to the square above. What can you tell me about the new number?



Give the students a few minutes to consider this situation before asking for a response.

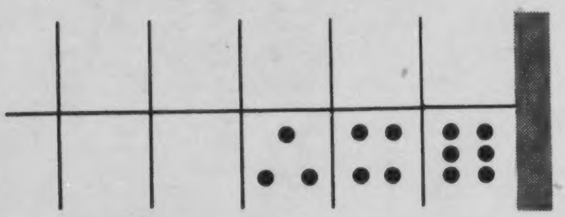
S: The number is 9 × 640.

S: 5,760; because 5,400 + 360 = 5,760 or 9 × 640 = 5,760.

Repeat this acitivity choosing numbers appropriate for the numerical level of your class. Emphasize that each time a checker is moved from a lower square to the square directly above it, the new number is nine times the original one.

Exercise 3

Display this configuration of checkers.

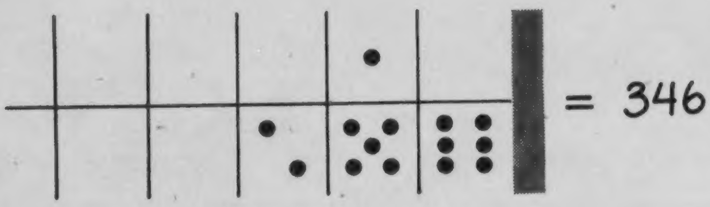


T: Write this number on your paper. (346)

Can you show the same number with one checker more?

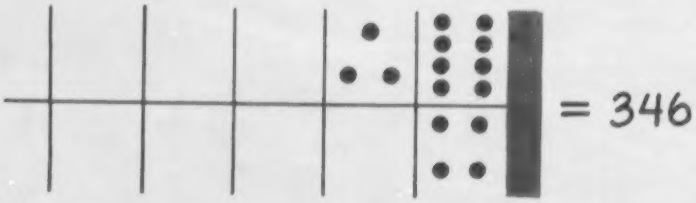
Invite a student to make a play. There are many possibilities; for example, a student might choose the "100 = 90 + 10" play.

S: 100 = 90 + 10.



Continue in this manner, asking for the same number using one, two, or three more checkers. Encourage students to make plays in which checkers always remain to the left of the bar.

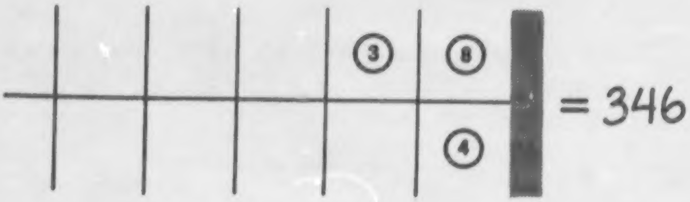
Continue this activity until the configuration shown below is obtained.



Display a set of weighted checkers (② - ⑩).

T: Who can make some trades using these checkers?

Let the students make trades until you have this configuration on the abacus.



T: Nabu has 346 crayons to put into nine boxes. He claims that by just looking at the abacus with 346 on it in this special way, he knows how many crayons to put into each box. On your paper, write the number of crayons in each box and how many crayons are left over.

Give the class a few minutes to think about the problem.

S: 38 crayons in each box and 4 crayons left over.

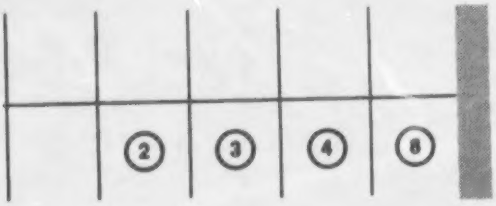
T: Very good. (Cover the ④-checker to call attention to only the checkers on the upper squares.) What is the number on the upper squares?

S: 9 x 38.

Write this sentence on the board.

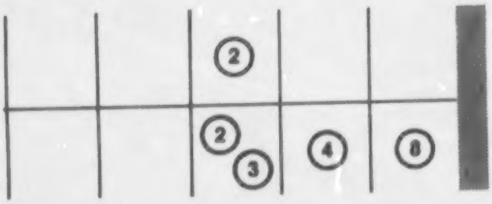
$$(9 \times 38) + 4 = 346$$

Display this configuration on the abacus.

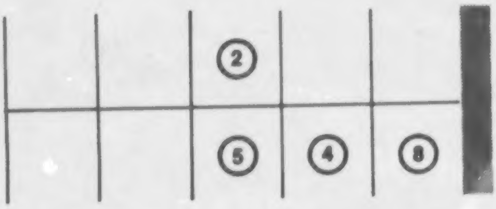


T: Which number is this? (2,348)

Can you show the same number using one more ②-checker?

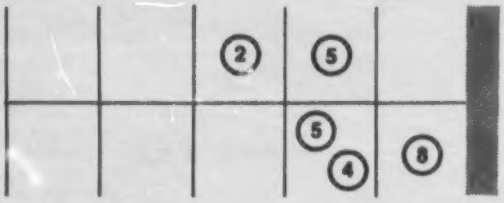


Exchange a ②-checker and a ③-checker for a ⑤-checker.

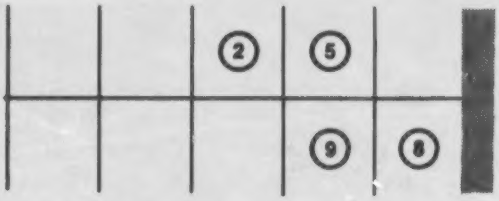


N 16

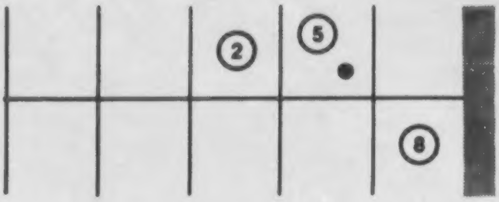
T: Can you show the same number using one more ⑤-checker?



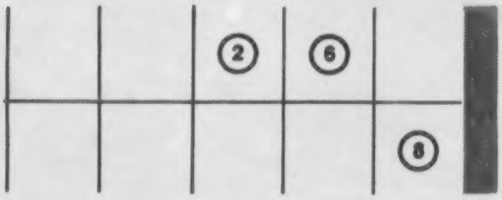
Exchange a ④-checker and a ⑤-checker for a ⑨-checker.



T: Can you trade the ⑨-checker for one regular checker?



Exchange a ⑤-checker and the regular checker for a ⑥-checker.



T: Nabu has 2,348 crayons to put into nine boxes. He claims that by just looking at the abacus with 2,348 on it in this special way, he knows how many crayons in each box and how many crayons left over.

Give the class a minute to think about the problem.

S: 260 crayons in each box and 8 crayons left over.

The class should conclude that (9 × 260) + 8 = 2,348.

Write this sentence on the board and ask the students to check the calculations.

N 20

4.4 Binary Code #2

CAPSULE LESSON SUMMARY

Review the rule of the binary abacus and Nabu's (binary) code for numbers. Practice writing numbers in the binary code and decoding numbers written in binary notation. Write addition and multiplication calculations in the binary system. Discover the binary code for $\frac{1}{3}$ and $\frac{1}{5}$.

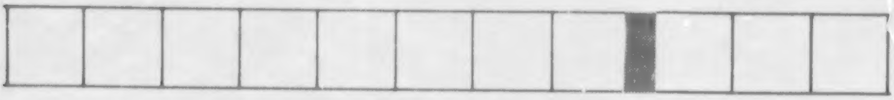
MATERIALS

Teacher: Colored chalk; demonstration Minicomputer checkers
Student: Paper; pencil

DESCRIPTION OF LESSON

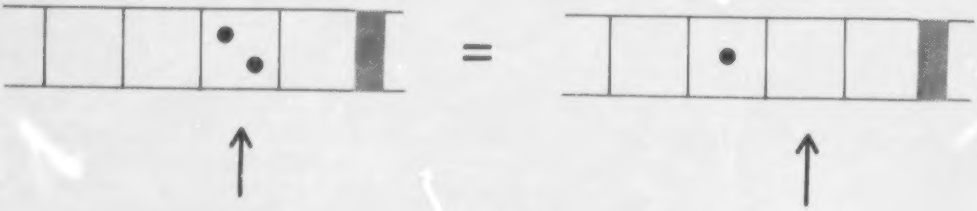
Exercise 1

Draw this part of an abacus on the chalkboard or use the Minicomputer boards.



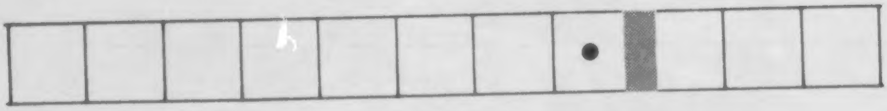
T: Do you remember the binary abacus and its rule?

Invite a student to explain the binary abacus rule to the class. The class should be reminded that the rule is:



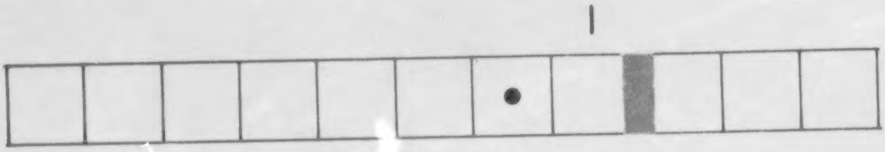
Two checkers on a board represents the same number as one checker on the next board to the left.

Put (draw) one checker on the abacus as shown here.



T: This is the number 1.

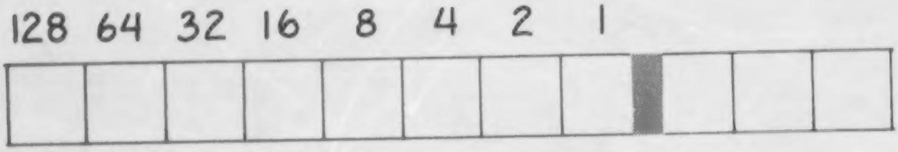
Label the board "1" and move the checker to the next board to the left.



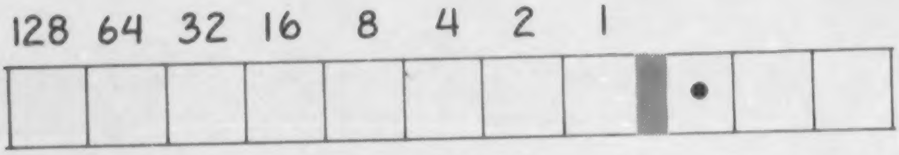
T: Which number is this?

S: 2.

Label the board "2" and continue in this manner until each board to the left of the bar is labeled.



Put one checker on the abacus as shown in the illustration below.



T: Which number is this?

To remind students who have difficulty with this situation, put another checker on the same board and ask:

T: Now which number is this?

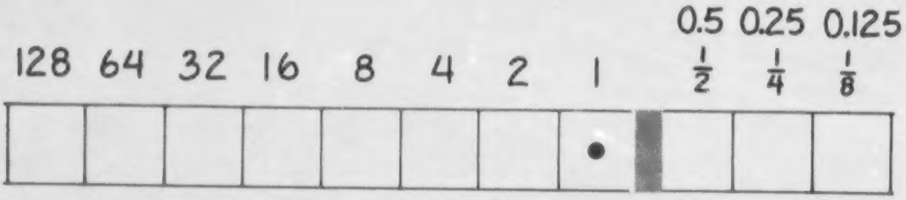
If necessary, ask a student to make a play to show that the number is 1.

S: 1.

T: Two checkers on this board is the number 1, so one checker on this board is which number?

S: 0.5 or $\frac{1}{2}$.

Label the board both "0.5" and "$\frac{1}{2}$". Continue this activity until each board to the right of the bar is also labeled.



Exercise 2

NOTE: You will notice that whenever we write numbers in the binary code, we use "**1**" and whenever we write numbers in the decimal code, we use "|". This device serves as a visual reminder to the students that we are writing in a code different from the decimal code; there is no mathematical reason for it.

Write this information on the board.

NABU'S CODE		DECIMAL WRITING
11	=	3
1011	=	11
1010	=	☐

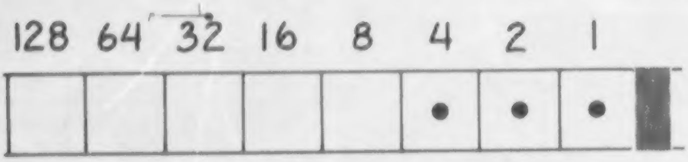
T: Do you remember Nabu's code? What number is this (point to the box)?

Allow students to whisper their guesses in your ear. Acknowledge aloud any correct guess and deny any incorrect one. For example, "Yes, that's correct." or "No, 8 is not the number." If no one suggests 10, do so yourself. Continue the activity with the problems below (answers are in the boxes). Tell students who need a hint that Nabu's code has something to do with the binary abacus.

NABU'S CODE		DECIMAL WRITING
1000	=	$\boxed{8}$
$\boxed{111}$	=	7
11001	=	$\boxed{25}$
101.1	=	$\boxed{5.5}$

For the benefit of students who have not yet cracked Nabu's code, do the following activity.

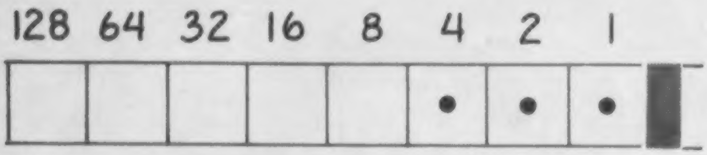
T: Who can put 7 on the binary abacus?



N 20

T: Now, who can write Nabu's code name for 7?

Ask a student to write the answer below the abacus.



NABU'S CODE DECIMAL WRITING

1 1 1 = 7

Continue the activity with these problems. If necessary, use the binary abacus as an aid.

$$11000 = \boxed{24}$$

$$\boxed{100011} = 35$$

$$1000110 = \boxed{70}$$

$$\boxed{101011} = 43$$

$$\boxed{11.01} = 3.25$$

$$111.001 = \boxed{7.125}$$

$$101.101 = \boxed{5.625}$$

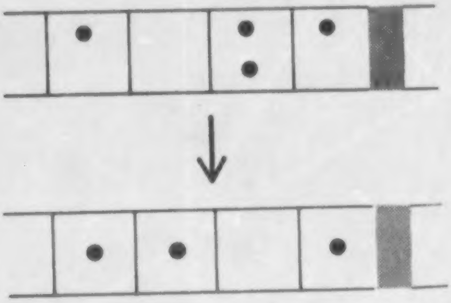
Exercise 3

Write this addition problem on the board.

$$\begin{array}{r} 1011 \\ +\quad 10 \\ \hline \end{array}$$

T: I have written an addition problem in the binary code (Nabu's code). How do you suggest we do this calculation?

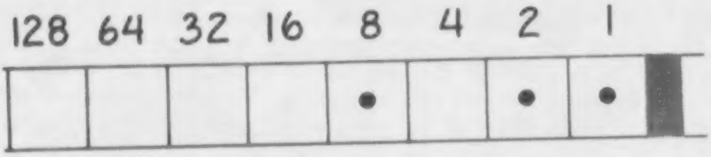
Some students may be able to solve the problem directly, while others will need the support of the binary abacus. See Exercise 3 of Lesson N 4.2, for a detailed description on how to relate plays on the abacus to the calculation. The next illustration summarizes the situation.



$$\begin{array}{r} 1011 \\ +\quad 10 \\ \hline 1101 \end{array}$$

Write this multiplication problem on the board and put the miltiplicand on the abacus.

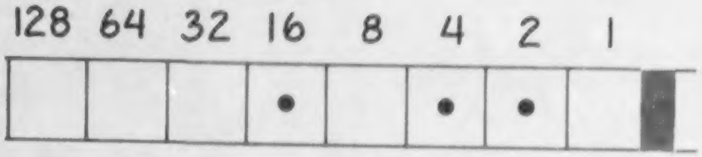
$$10 \times 1011 = \boxed{}$$



T: I wrote a multiplication problem in the binary code. Who can perform the calculation very quickly?

N 20

S: The number on the abacus is being multiplied by 2, so move each checker one square to the left.

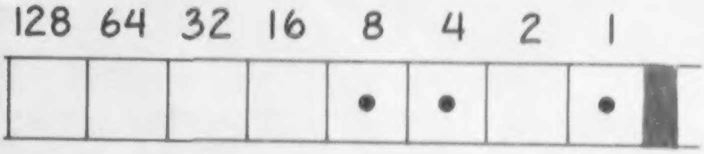


T: How should we write the product in binary code?

$$10 \times 1011 = \boxed{10110}$$

Write this calculation on the board and put the multiplicand on the abacus.

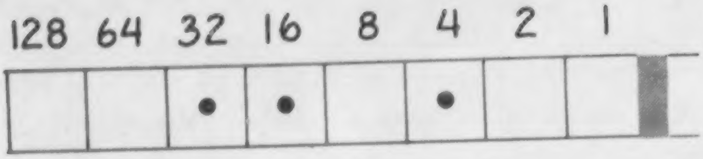
$$100 \times 1101 = \boxed{}$$



T: How can we perform this calculation quickly?

S: We are multiplying by 4, so move each checker two boards to the left.

S: Just add two "0"'s to "1101".



1 1 0 1 0 0

$$100 \times 1101 = \boxed{110100}$$

Continue the activity with two more similar calculations.

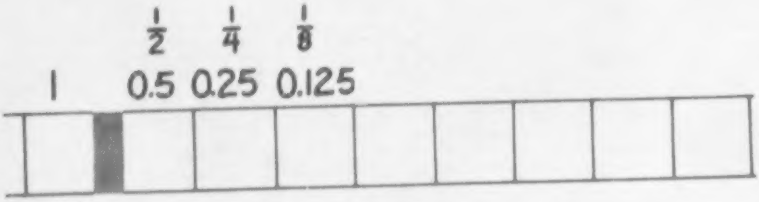
$$1000 \times 1101 = \boxed{1101000}$$

$$10000 \times 1101 = \boxed{11010000}$$

For each calculation, the students should first realize that the multiplication is by 8 (16) and that the calculation may be performed quickly by moving each checker on the abacus three (four) boards to the left.

Exercise 4

Display this part of the binary abacus.

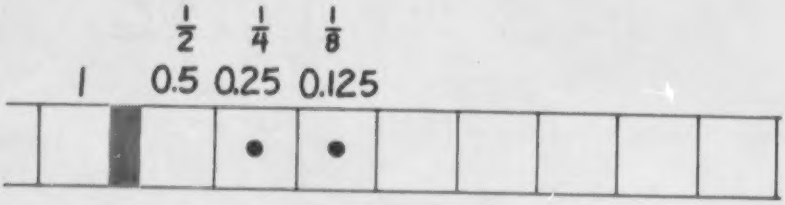


N 20

T: We know how to put $\frac{1}{2}$ and $\frac{1}{4}$ on the binary abacus. How do we put $\frac{1}{3}$ on the abacus?

Let the class discuss this situation. Try to pursue suggestions as much as possible, but discourage wild inventions. In the following dialogue, we describe how the solution of this problem developed in one of our development classes.

S: Put $\frac{1}{4}$ and $\frac{1}{8}$ on the abacus.



T: Look at the decimal names for $\frac{1}{4}$ and $\frac{1}{8}$. Which number is this?

S: 0.375.

Write the calculation on the board.

$$0.25 + 0.125 = 0.375$$

T: Do you remember a decimal name for $\frac{1}{3}$?

S: "0.333333..." or "$0.\overset{*}{3}$".

Write the equation on the board for emphasis.

$$0.25 + 0.125 = 0.375$$

$$\frac{1}{3} = 0.333333\ldots = 0.\overset{*}{3}$$

T: Is the number on the abacus $\frac{1}{3}$?

S: No, it is larger than $\frac{1}{3}$.

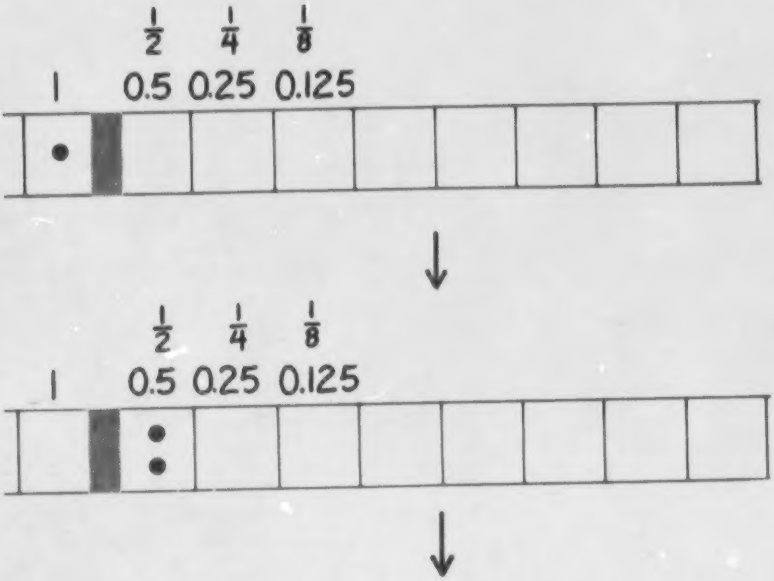
T: Why is it so easy to put $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ on the abacus?

S: Because it is easy to put 1 on the abacus with two, four, or eight checkers on the same square.

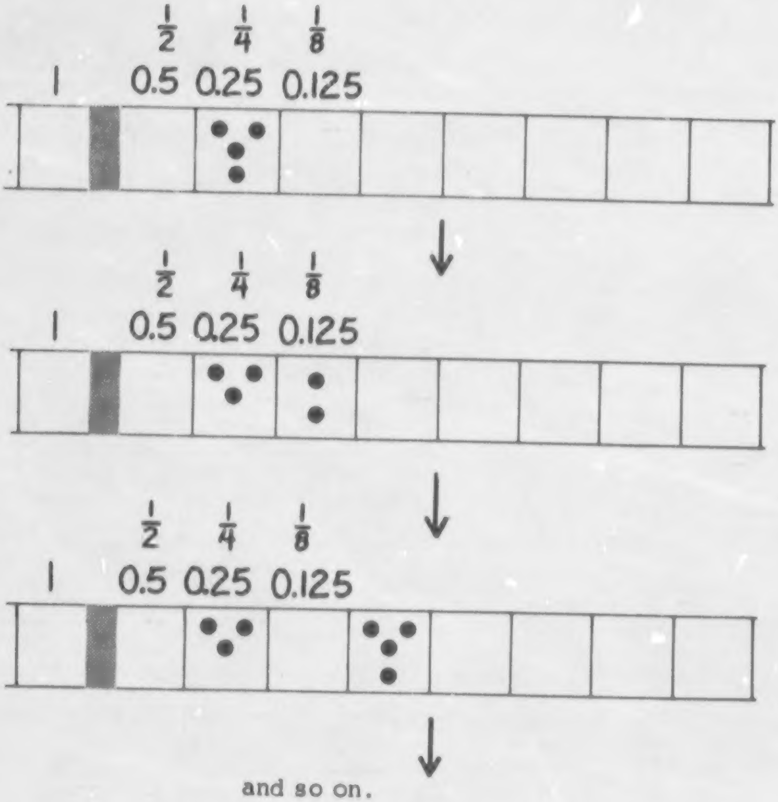
T: Let me suggest a way to get $\frac{1}{3}$ on the abacus. Let's start with 1.

T: Now, let's put 1 on the abacus another way. We would like to get three checkers on any board with checkers on it so that it will be easy to get $\frac{1}{3}$. Which plays should we make?

Students should suggest making the following sequence of plays. If necessary, make the first few plays yourself and when students discover the pattern, let them make the plays.

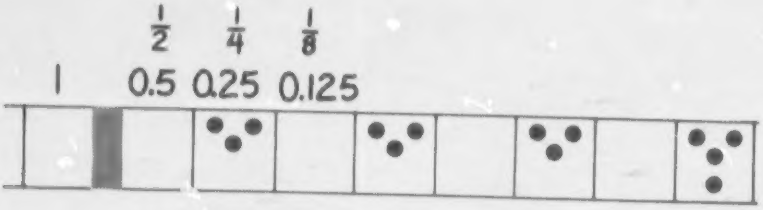


N 20


$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$
1 0.5 0.25 0.125
$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$
1 0.5 0.25 0.125
$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$
1 0.5 0.25 0.125

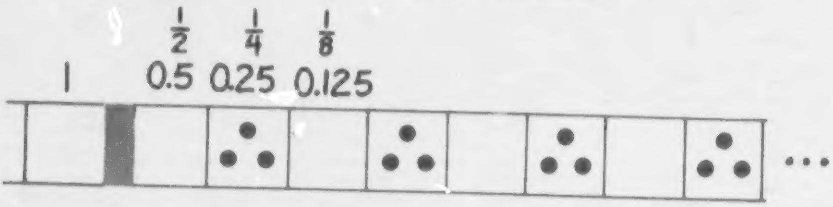
and so on.

Continue until this configuration is on the abacus.


$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$
1 0.5 0.25 0.125

T: Will this ever end?

S: No, we can go on and on making backward plays.

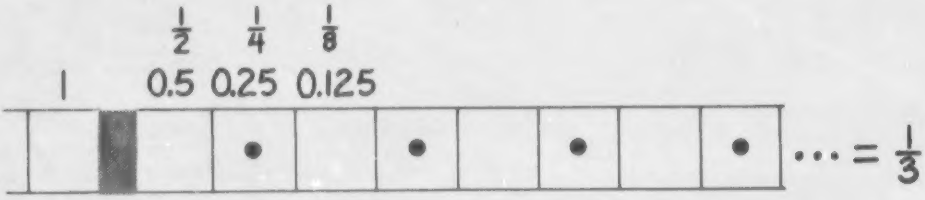

$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$
1 0.5 0.25 0.125
...

T: Which number is on the abacus?

S: 1.

T: Now, who can show $\frac{1}{3}$?

Let a student remove two out of every three checkers, leaving this configuration on the abacus.



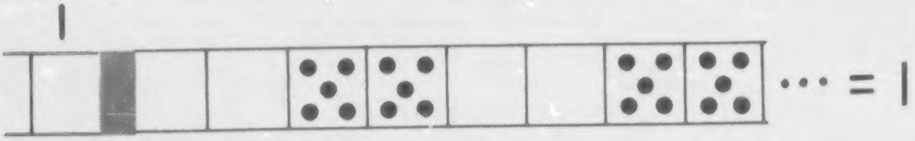
T: Who can write a name for $\frac{1}{3}$ using Nabu's code? Write the name on your paper.

Give the class a few minutes to work and then ask a student to write Nabu's code for $\frac{1}{3}$ on the board.

NABU'S CODE	DECIMAL WRITING

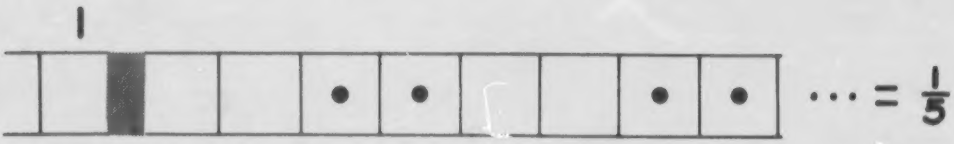
$$0.01010101\cdots = 0.\overset{*}{\overline{01}} = \frac{1}{3} = 0.333\cdots = 0.\overset{*}{3}$$

If some students are interested in this activity and time permits, challenge them to find Nabu's code for $\frac{1}{5}$. The strategy is the same as for $\frac{1}{3}$; put 1 on the abacus and then make backward plays to get five checkers on all squares which have checkers on them.



N 20

Then remove four of every five checkers, leaving this configuration.



NABU'S CODE

DECIMAL WRITING

$$0.001100110011\cdots = 0.\overset{*}{\overline{0011}} = \frac{1}{5} = 0.2$$

5

STANDARD ALGORITHMS OF ARITHMETIC

The lessons in this chapter review the standard algorithms for addition, subtraction, and multiplication by presenting puzzle type problems which reinforce understanding of the algorithmic processes in each case. By extending earlier division experiences (see the chapters on division in Parts I, II, and III of the CSMP Intermediate Grades Program), the lessons present a form of the standard division algorithm. Puzzles involving the division algorithm again reinforce understanding of the algorithmic process.

CAPSULE LESSON SUMMARIES

Section 5.1:	Review the standard algorithms for addition, subtraction, and multiplication. Solve some puzzle type problems involving these algorithms. (Worksheets N50, N51, N52)
Section 5.2:	Solve division problems suggested by situations in which a given number of bottles are packed into boxes, each box holding a specified number of bottles. Present a form of the standard division algorithm.
Section 5.3:	Solve division problems suggested by situations in which a given number of bottles are packed into boxes, each box holding a specified number of bottles. Present a form of the standard division algorithm. Individually, solve some puzzles involving the division algorithm. (Worksheets N53, N54, N55)

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5.1 Algorithm Puzzles

CAPSULE LESSON SUMMARY

Review the standard algorithms for addition, subtraction, and multiplication. Solve some puzzle type problems involving these algorithms.

MATERIALS

Teacher: Chalk
Student: Paper; pencil; Worksheets N50, N51, N52

DESCRIPTION OF LESSON

Exercise 1

Write these two problems on the board and ask the students to copy and solve them. Answers are in the boxes.

$$\begin{array}{r} 3{,}782 \\ +\,2{,}659 \\ \hline \boxed{6{,}441} \end{array} \qquad \begin{array}{r} 678 \\ 349 \\ +\,275 \\ \hline \boxed{1{,}302} \end{array}$$

Check the problems collectively by asking two students to solve the problems and to explain the calculations at the board.

T: I had two more problems for you, but an eraser gremlin erased part of my calculations. I will put boxes in place of some of the numerals. Try to find out which problem I had written.

Write these problems on the board and give the students a few minutes to solve them independently.

$$\begin{array}{rcccc} & & 6 & \square & 3 \\ + & & 9 & 4 & \square \\ \hline & \square & \square & 2 & 0 \end{array} \qquad \begin{array}{rcccc} & 2 & \square & 7 & \square \\ + & \square & 7 & \square & 8 \\ \hline & 7 & 5 & 1 & 3 \end{array}$$

$$\begin{array}{rccccc} & & 4 & \square & 7 & 5 \\ & & \square & 3 & 4 & \square \\ + & & 5 & 7 & \square & 2 \\ \hline & 1 & 3 & 4 & 0 & 3 \end{array}$$

When most of the class have found solutions, invite several students to the board to complete the problems and, if necessary, to explain their solutions. Solutions are given below.

$$\begin{array}{rcccc} & & 1 & 1 & \\ & & 6 & \boxed{7} & 3 \\ + & & 9 & 4 & \boxed{7} \\ \hline & \boxed{1} & \boxed{6} & 2 & 0 \end{array} \qquad \begin{array}{rcccc} & 1 & 1 & 1 & \\ & 2 & \boxed{7} & 7 & \boxed{5} \\ + & \boxed{4} & 7 & \boxed{3} & 8 \\ \hline & 7 & 5 & 1 & 3 \end{array}$$

$$\begin{array}{rccccc} & & 1 & 2 & 1 & \\ & & 4 & \boxed{2} & 7 & 5 \\ & & \boxed{3} & 3 & 4 & \boxed{6} \\ + & & 5 & 7 & \boxed{8} & 2 \\ \hline & 1 & 3 & 4 & 0 & 3 \end{array}$$

N 6

Exercise 2

Write these two problems on the board and ask the students to solve them on their papers. Answers are in the boxes.

$$\begin{array}{r} 982 \\ -\,545 \\ \hline \boxed{437} \end{array} \qquad \begin{array}{r} 3{,}428 \\ -\,1{,}475 \\ \hline \boxed{1{,}953} \end{array}$$

Check the problems collectively by letting students explain the calculations at the board.

Write these problems on the board and ask the students to copy them.

$$\begin{array}{r} \square\,7\,\square\,6 \\ -\,4\,\square\,5\,\square \\ \hline 5\,5\,3\,4 \end{array} \qquad \begin{array}{r} 8\,\square\,9\,2 \\ -\,\square\,7\,\square\,\square \\ \hline 3\,6\,5\,6 \end{array}$$

$$\begin{array}{r} \square\,8\,2\,\square \\ -\,6\,\square\,\square\,5 \\ \hline 3\,3\,4\,8 \end{array}$$

T: I have written three more problems for you on the board. The boxes are for numerals that the eraser gremlin erased. Remember, exactly one numeral can be written in each box.

After several minutes of individual work, check the problems collectively. Answers are provided below.

$$\begin{array}{r} \boxed{9}\;7\;\boxed{8}\;6 \\ -\;4\;\boxed{2}\;5\;\boxed{2} \\ \hline 5\;5\;3\;4 \end{array} \qquad \begin{array}{r} 8\;\boxed{3}\;9\;2 \\ -\;\boxed{4}\;7\;\boxed{3}\;\boxed{6} \\ \hline 3\;6\;5\;6 \end{array}$$

$$\begin{array}{r} \boxed{9}\;8\;2\;\boxed{3} \\ -\;6\;\boxed{4}\;\boxed{7}\;5 \\ \hline 3\;3\;4\;8 \end{array}$$

Exercise 3

Write these two problems on the board and ask the students to solve them on their papers. Answers are in the boxes.

$$\begin{array}{r} 2{,}376 \\ \times\quad 4 \\ \hline \boxed{9{,}504} \end{array} \qquad \begin{array}{r} 2{,}483 \\ \times\quad 23 \\ \hline 7449 \\ 49660 \\ \hline \boxed{57{,}109} \end{array}$$

$$\begin{array}{r} 3{,}576 \\ \times\ 134 \\ \hline 14304 \\ 107280 \\ 357600 \\ \hline \boxed{479{,}184} \end{array}$$

N 6

Check the problems collectively by letting students explain the calculations at the board.

Write these problems on the board and ask the students to solve them.

$$\begin{array}{r} 2\square6\square \\ \times\qquad 3 \\ \hline \square1\square1 \end{array} \qquad\qquad \begin{array}{r} 3\square5\square \\ \times\qquad 7 \\ \hline \square\square2\square6 \end{array}$$

After several minutes of individual work, check the problems collectively. Invite several students to the board to complete the problems and, if necessary, to explain their solutions. Solutions are provided below.

$$\begin{array}{r} 2\boxed{3}6\boxed{7} \\ \times\qquad 3 \\ \hline \boxed{7}1\boxed{0}1 \end{array} \qquad\qquad \begin{array}{r} 3\boxed{4}5\boxed{8} \\ \times\qquad 7 \\ \hline \boxed{2}\boxed{4}2\boxed{0}6 \end{array}$$

Worksheets N50 , N51, and N52 are available.

Name____________________ N 50 *

Complete each problem. Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each box to make the calculations correct. You may use a digit more than once.

$$\begin{array}{r} 3{,}789 \\ +\,4{,}625 \\ \hline 8{,}414 \end{array} \qquad \begin{array}{r} 4{,}567 \\ 3{,}285 \\ +\,1{,}848 \\ \hline 9{,}700 \end{array}$$

$$\begin{array}{r} \boxed{4}\,7\,\boxed{5}\,6 \\ +\,2\,8\,2\,\boxed{6} \\ \hline 7\,\boxed{5}\,8\,2 \end{array} \qquad \begin{array}{r} \boxed{4}\,5\,6\,\boxed{7} \\ +\,2\,\boxed{4}\,8\,7 \\ \hline 7\,0\,\boxed{5}\,4 \end{array}$$

$$\begin{array}{r} 2\,\boxed{8}\,3\,7 \\ 4\,4\,4\,\boxed{3} \\ +\,1\,2\,\boxed{1}\,5 \\ \hline \boxed{8}\,4\,9\,5 \end{array}$$

Name____________________ N 51 **

Complete each problem. Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each box to make the calculations correct. You may use a digit more than once.

$$\begin{array}{r} 9{,}826 \\ -\,4{,}352 \\ \hline 5{,}474 \end{array} \qquad \begin{array}{r} 7{,}349 \\ -\,4{,}856 \\ \hline 2{,}493 \end{array}$$

$$\begin{array}{r} \boxed{9}\,8\,\boxed{7}\,2 \\ -\,3\,\boxed{6}\,2\,\boxed{4} \\ \hline 6{,}2\,4\,8 \end{array} \qquad \begin{array}{r} \boxed{4}\,2\,\boxed{3}\,1 \\ -\,2\,\boxed{5}\,7\,\boxed{6} \\ \hline 1{,}6\,5\,5 \end{array}$$

Name____________________ N 52 ***

Complete each problem. Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each box to make the calculations correct. You may use a digit more than once.

$$\begin{array}{r} 6{,}437 \\ \times\ 35 \\ \hline 32{,}185 \\ 193{,}110 \\ \hline 225{,}295 \end{array} \qquad \begin{array}{r} 3\,\boxed{5}\,7\,\boxed{6} \\ \times\ 9 \\ \hline \boxed{3}\,\boxed{2}\,1\,\boxed{8}\,4 \end{array}$$

$$\begin{array}{r} \boxed{4}\,6\,\boxed{5} \\ \times\ \boxed{3}\,7 \\ \hline \boxed{3}\,2\,\boxed{5}\,5 \\ 1\,\boxed{3}\,9\,5\,0 \\ \hline \boxed{1}\,\boxed{7}\,\boxed{2}\,\boxed{0}\,\boxed{5} \end{array} \qquad \begin{array}{r} \boxed{2}\,9\,7 \\ \times\ 3\,\boxed{1}\,6 \\ \hline 1\,7\,\boxed{8}\,2 \\ 2\,9\,7\,0 \\ \boxed{8}\,9\,\boxed{1}\,0\,0 \\ \hline 9\,\boxed{3}\,8\,5\,\boxed{2} \end{array}$$

N 13

5.2 Division Algorithm #1

CAPSULE LESSON SUMMARY

Solve division problems suggested by situations in which a given number of bottles are packed into boxes, each box holding a specified number of bottles. Present a form of the standard division algorithm.

MATERIALS

Teacher: Chalk
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

Draw this table on the board.

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
70	6		

Point to the appropriate places in the table as you describe the problem.

T: Nabu collects bottles for a recycling plant. One day he collected 70 bottles. He must put the bottles into boxes and each box holds 6 bottles. How many boxes can he fill completely and how many bottles will be left over?

S: 11 boxes will be filled, because 11 × 6 = 66.

S: 4 bottles will be left over.

Add this information to your table

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
70	6	11	4

Continue in the same way for the information in the table below. Feel free to adjust the level of difficulty to the abilities of your class. Correct answers are in the boxes.

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
70	6	11	4
134	10	$\boxed{13}$	$\boxed{4}$
2,543	10	$\boxed{254}$	$\boxed{3}$
205	8	$\boxed{25}$	$\boxed{5}$
500	20	$\boxed{25}$	$\boxed{0}$
3,628	100	$\boxed{36}$	$\boxed{28}$
100	7	$\boxed{14}$	$\boxed{2}$

Exercise 2

T: Suppose we had 3,649 bottles to put into boxes of 24. Would we need as many as 100 boxes?

S: We would need more than 100 boxes, because 100 × 24 = 2,400.

N 13

T: Would we need as many as 200 boxes?

S: We would need less than 200 boxes, because 200 × 24 = 4,800.

T: So we would need between 100 and 200 boxes. Let's fill 100 boxes.

Show the filling of 100 boxes by writing this calculation on the board.

$$\begin{array}{r l} 3{,}6\,4\,9 & \\ -2{,}4\,0\,0 & \textbf{100}\text{ boxes} \\ \hline 1{,}2\,4\,9 & \end{array}$$

T: If we fill 100 boxes, how many bottles will be left over?

S: 1,249.

T: Now we must put 1,249 bottles into boxes of 24. About how many boxes will we need?

S: We can put 960 bottles into 40 boxes. (40 × 24 = 960).

S: We need more than 50 boxes, because 50 × 24 = 1,200.

S: We need between 50 and 60 boxes, because 60 × 24 = 1,440.

T: Let's fill 50 boxes.

Show the filling of 50 boxes by writing this calculation on the board.

$$\begin{array}{r l} 3{,}6\,4\,9 & \\ -\ 2{,}4\,0\,0 & \textbf{100}\text{ boxes} \\ \hline 1{,}2\,4\,9 & \\ -\ 1{,}2\,0\,0 & \textbf{50}\text{ boxes} \\ \hline 4\,9 & \end{array}$$

T: How many bottles are left over?

S: 49 bottles.

T: How many boxes of 24 do we need for 49 bottles?

S: 2 boxes, because 2 × 24 = 48 and 1 bottle left over.

Record this information on the board.

$$\begin{array}{rl} 3,649 & \\ -\,2,400 & 100 \text{ boxes} \\ \hline 1,249 & \\ -\,1,200 & 50 \text{ boxes} \\ \hline 49 & \\ -\qquad 48 & 2 \text{ boxes} \\ \hline 1 & \end{array}$$

T: How many boxes of 24 are needed for 3,649 bottles? How many are left over?

S: 152 boxes and 1 bottle left over.

$$\begin{array}{rl} 3,649 & \\ -\,2,400 & 100 \text{ boxes} \\ \hline 1,249 & \\ -\,1,200 & 50 \text{ boxes} \\ \hline 49 & \\ -\qquad 48 & 2 \text{ boxes} \\ \hline 1 & 152 \text{ boxes} \end{array}$$

N 13

Repeat this activity with putting 8,785 bottles into boxes of 36.

$$\begin{array}{rr} 8{,}785 & \\ -7{,}200 & 200 \text{ boxes} \\ \hline 1{,}585 & \\ -1{,}440 & 40 \text{ boxes} \\ \hline 145 & \\ -144 & 4 \text{ boxes} \\ \hline 1 & 244 \text{ boxes} \end{array}$$

T: This problem of putting 8,785 bottles into boxes of 36 is a division problem. This is a way to write the calculation.

Write this calculation on the board next to your solution of the bottles and boxes problem.

$$\begin{array}{rr|r} & 244 & R=1 \\ 36 \big) & 8{,}785 & \\ & -7{,}200 & 200 \\ \hline & 1{,}585 & \\ & -1{,}440 & 40 \\ \hline & 145 & \\ & -144 & 4 \\ \hline & 1 & 244 \end{array}$$

Repeat this activity by solving the problem given below, either collectively or independently.

$$42\,\overline{)\,9{,}874}$$

If you choose to have the students work independently, check the problems collectively. The solution is given below.

$$\begin{array}{r|r} \multicolumn{1}{r}{235 \quad R=4} & \\ 42\,\overline{)\,9{,}874} & \\ -8{,}400 & 200 \\ \hline 1{,}474 & \\ -1{,}260 & 30 \\ \hline 214 & \\ -\quad 210 & 5 \\ \hline 4 & 235 \end{array}$$

Exercise 3

Write these problems on the board and ask the students to solve them independently.

$12\,\overline{)\,3{,}878.}$ $\qquad$ $15\,\overline{)\,9{,}705}$

$24\,\overline{)\,5{,}785}$ $\qquad$ $27\,\overline{)\,8{,}966}$

Check the problems collectively by inviting students to explain the calculations at the board.

N 13

$$\begin{array}{rr|r}
 & 323 & \text{R}=2 \\ \cline{2-2}
12) & 3{,}878 & \\
 & -3{,}600 & 300 \\ \hline
 & 278 & \\
 & -\ \ 240 & 20 \\ \hline
 & 38 & \\
 & -\ \ \ \ 36 & 3 \\ \hline
 & 2 & 323
\end{array}$$

$$\begin{array}{rr|r}
 & 647 & \text{R}=0 \\ \cline{2-2}
15) & 9{,}705 & \\
 & -9{,}000 & 600 \\ \hline
 & 705 & \\
 & -\ \ 600 & 40 \\ \hline
 & 105 & \\
 & -\ \ 105 & 7 \\ \hline
 & 0 & 647
\end{array}$$

$$\begin{array}{rr|r}
 & 241 & \text{R}=1 \\ \cline{2-2}
24) & 5{,}785 & \\
 & -4{,}800 & 200 \\ \hline
 & 985 & \\
 & -\ \ 960 & 40 \\ \hline
 & 25 & \\
 & -\ \ \ \ 24 & 1 \\ \hline
 & 1 & 241
\end{array}$$

$$\begin{array}{rr|r}
 & 332 & \text{R}=2 \\ \cline{2-2}
27) & 8{,}966 & \\
 & -8{,}100 & 300 \\ \hline
 & 866 & \\
 & -\ \ 810 & 30 \\ \hline
 & 56 & \\
 & -\ \ \ \ 54 & 2 \\ \hline
 & 2 & 332
\end{array}$$

5.3 Division Algorithm #2

CAPSULE LESSON SUMMARY

Solve division problems suggested by situations in which a given number of bottles are packed into boxes, each box holding a specifed number of bottles. Present a form of the standard division algorithm. Individually, solve some puzzles involving the division algorithm.

MATERIALS

Teacher: Chalk
Student: Paper; pencil; Worksheets N53, N54, N55

DESCRIPTION OF LESSON

Begin the lesson with a brief mental arithmetic activity using some division problems such as the ones suggested here. Feel free to adjust the problems to suit the numerical abilities of your students. Answers are in the boxes.

$21 \div 3 = \boxed{7}$;
$210 \div 3 = \boxed{70}$;
$2{,}100 \div 3 = \boxed{700}$;
$2{,}100 \div 30 = \boxed{70}$;
$2{,}100 \div 300 = \boxed{7}$;

$16 \div 4 = \boxed{4}$;
$160 \div 4 = \boxed{40}$;
$1{,}600 \div 4 = \boxed{400}$;
$1{,}600 \div 40 = \boxed{40}$;
$1{,}600 \div 400 = \boxed{4}$;

$42 \div 6 = \boxed{7}$;
$420 \div 6 = \boxed{70}$;
$4{,}200 \div 6 = \boxed{700}$;
$4{,}200 \div 60 = \boxed{70}$;
$4{,}200 \div 600 = \boxed{7}$;

$56 \div 7 = \boxed{8}$
$560 \div 7 = \boxed{80}$
$5{,}600 \div 7 = \boxed{800}$
$5{,}600 \div 70 = \boxed{80}$
$5{,}600 \div 700 = \boxed{8}$

Exercise 1

Draw this table on the board.

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
98	6		

N 26

Point to the appropriate columns of the table as you describe the problem.

T: Nabu collects bottles for a recycling plant. One day he collected 98 bottles. He must put these bottles into boxes; each box holds 6 bottles. How many boxes can he fill completely and how many bottles will be left over?

S: 16 boxes can be filled, because 16 × 6 = 96.

S: 2 bottles will be left over.

Add this information to your table.

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
98	6	16	2

Continue using the information in the table below and asking students to provide answers in the boxes Feel free to adjust the level of difficulty to the numerical abilities of your class.

Number of Bottles	Number of Bottles in Each Box	Number of Boxes Completely Filled	Number of Bottles Left Over
98	6	16	2
98	12	$\boxed{8}$	$\boxed{2}$
103	8	$\boxed{12}$	$\boxed{7}$
157	21	$\boxed{7}$	$\boxed{10}$
578	60	$\boxed{9}$	$\boxed{38}$
3,548	100	$\boxed{35}$	$\boxed{48}$
5,280	200	$\boxed{26}$	$\boxed{80}$

Exercise 2

T: Suppose we have 3,587 bottles to put into boxes of 26. Would we need as many as 100 boxes?

S: Yes; 100 × 26 = 2,600.

T: Would we need as many as 200 boxes?

S: No; 200 × 26 = 5,200.

T: So we need between 100 and 200 boxes. Let's fill 100 boxes.

Show the filling of 100 boxes by writing this calculation on the board.

$$\begin{array}{rl} 3{,}587 & \\ -\ 2{,}600 & \text{100 boxes} \\ \hline 987 & \end{array}$$

T: If we fill 100 boxes, how many bottles will be left over?

S: 987.

T: We must put 987 bottles into boxes of 26. About how many boxes will we need? 10?

S: We need many more than 10, because 10 × 26 = 260.

T: How about 20?

S: 20 × 26 = 520; so we need more than 20.

S: Between 30 and 40 boxes, because 30 × 26 = 780 and 40 × 26 = 1,040.

T: Let's fill 30 boxes.

Show the filling of 30 boxes by writing this calculation on the board.

$$
\begin{array}{rrl}
 & 3{,}587 & \\
- & 2{,}600 & 100 \text{ boxes} \\
\hline
 & 987 & \\
- & 780 & 30 \text{ boxes} \\
\hline
 & 207 &
\end{array}
$$

T: How many bottles are left over?

S: 207 bottles.

T: How many boxes of 26 do we need for 207 bottles?

S: 7 boxes, because 7 × 26 = 182 and 25 bottles are left over.

Record this information in the calculation on the board.

T: How many boxes of 26 are needed for 3,587 bottles? How many bottles are left over?

S: 137 boxes with 25 bottles left over.

$$
\begin{array}{rrl}
 & 3{,}587 & \\
- & 2{,}600 & 100 \text{ boxes} \\
\hline
 & 987 & \\
- & 780 & 30 \text{ boxes} \\
\hline
 & 207 & \\
- & 182 & 7 \text{ boxes} \\
\hline
 & 25 & 137 \text{ boxes}
\end{array}
$$

T: This problem of putting 3,587 bottles into boxes of 26 is a division problem. This is a way to write the calculation.

Amend the calculation on the board to indicate that it is the calculation for a division problem.

$$\begin{array}{r r | r}
 & 137 \quad R=25 & \\
\cline{2-2}
26 \,\big) & 3{,}587 & \\
 & -2{,}600 & 100 \\
\cline{2-3}
 & 987 & \\
 & -780 & 30 \\
\cline{2-3}
 & 207 & \\
 & -182 & 7 \\
\cline{2-3}
 & 25 & 137
\end{array}$$

Repeat this activity by solving another division problem such as the one below, either collectively or independently.

$$32 \,\big)\overline{\;6{,}536\;}$$

If you choose to have the students work independently, check the problem collectively. The solution is given below.

$$\begin{array}{r r | r}
 & 204 \quad R=8 & \\
\cline{2-2}
32 \,\big) & 6{,}536 & \\
 & -6{,}400 & 200 \\
\cline{2-3}
 & 136 & \\
 & -128 & 4 \\
\cline{2-3}
 & 8 & 204
\end{array}$$

N 26

Write these division problems on the board and ask the students to solve them independently. Solutions are provided here in the boxes.

$$\begin{array}{r} \boxed{3\ 1\ 2 \quad \text{R}=2} \\ 14 \overline{\smash{)}\,4,3\ 7\ 0} \end{array} \qquad \begin{array}{r} \boxed{5\ 4\ 3 \quad \text{R}=0} \\ 18 \overline{\smash{)}\,9,7\ 7\ 4} \end{array}$$

$$\begin{array}{r} \boxed{2\ 0\ 8 \quad \text{R}=12} \\ 24 \overline{\smash{)}\,5,0\ 0\ 4} \end{array} \qquad \begin{array}{r} \boxed{2\ 4 \quad \text{R}=5} \\ 115 \overline{\smash{)}\,2,7\ 6\ 5} \end{array}$$

After a short while, check the problems collectively by asking students to explain the calculations at the board.

Worksheets N53, N54, and N55 are available.

Name ____________________ N 53 *

Complete each problem. Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each box to make the calculations correct. You may use a digit more than once.

```
      1 1 7     R=1
15 ) 1,7 5 6 |
    -1,5 0 0 | 100
        2 5 6 |
    -   1 5 0 |  10
        1 0 6 |
    -   1 0 5 |   7
            1 | 117
```

```
      1 5   R=2
 4 )  6 2 |
     -4 0 | 1 0
      2 2 |
     -2 0 |   5
        2 | 1 5
```

```
      2 3   R=2
 5 ) 1 1 7 |
    -1 0 0 | 2 0
       1 7 |
    -  1 5 |   3
         2 | 2 3
```

```
       3 2 1   R=8
21 ) 6,7 4 9 |
    -6,3 0 0 | 300
        4 4 9 |
      - 4 2 0 |  20
          2 9 |
        - 2 1 |   1
            8 | 321
```

Name ____________________ N 54 **

Complete each problem. Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each box to make the calculations correct. You may use a digit more than once.

```
      2 3   R=4
 7 ) 1 6 5 |
    -1 4 0 | 2 0
       2 5 |
    -  2 1 |   3
         4 | 2 3
```

```
      3 4   R=8
15 ) 5 1 8 |
    -4 5 0 | 3 0
       6 8 |
    -  6 0 |   4
         8 | 3 4
```

```
      6 7 4   R=0
 8 ) 5 3 9 2 |
    -4 8 0 0 | 6 0 0
       5 9 2 |
    -  5 6 0 |   7 0
         3 2 |
    -    3 2 |     4
           0 | 6 7 4
```

```
     2 3 0 6   R=1
 4 ) 9 2 2 5 |
    -8 0 0 0 | 2 0 0 0
     1 2 2 5 |
    -1 2 0 0 |   3 0 0
         2 5 |
    -    2 4 |       6
           1 | 2 3 0 6
```

Name ____________________ N 55 ***

Complete each problem. Put one of the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in each box to make the calculations correct. You may use a digit more than once.

```
      4 9 3   R=2
 9 ) 4 4 3 9 |
    -3 6 0 0 | 4 0 0
       8 3 9 |
    -  8 1 0 |   9 0
         2 9 |
    -    2 7 |     3
           2 | 4 9 3
```

```
        7 8   R=2 0
23 ) 1 8 1 4 |
    -1 6 1 0 | 7 0
       2 0 4 |
    -  1 8 4 |   8
         2 0 | 7 8
```

```
        2 9 4   R=3 4
57 ) 1 6 7 9 2 |
     1 1 4 0 0 | 2 0 0
       5 3 9 2 |
    -  5 1 3 0 |   9 0
         2 6 2 |
    -    2 2 8 |     4
           3 4 | 2 9 4
```

6

COMPOSITION

In this chapter, composition of numerical functions is reviewed in various arrow diagram settings. Attention is focused on the distinction between composites which involve only functions of the form "$\square\times$" or "$\div\square$", or which involve only functions of the form "$+\square$" or "$-\square$", and composites which involve some combination of multiplication, division, addition, and subtraction functions. Notice that

$$3\times \circ \div 4 = \frac{3}{4}\times = \div 4 \circ 3\times$$

and

$$+3 \circ +4 = +7 = +4 \circ +3,$$

whereas

$$3\times \circ +4 \neq +4 \circ 3\times.$$

The experiences of this chapter lead one to observe, for example, that

$$+4 \circ 3\times = 3\times \circ +12$$

and

$$+12 \circ \div 2 = \div 2 \circ +6$$

and

$$-4 \circ 3\times = 3\times \circ -12.$$

CAPSULE LESSON SUMMARIES

Section 6.1: Label arrows which complete a cycle of addition and subtraction arrows or of multiplication and division arrows. Use the hand-calculator as support for the situation. Using compositions of addition and multiplication functions, label arrows in pictures

such as

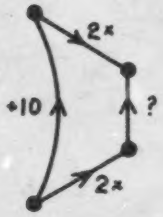
$$\begin{tikzcd}
\bullet \arrow[r, "2\times"] & \bullet \\
\bullet \arrow[u, "+4"] \arrow[r, "2\times"'] & \bullet \arrow[u, "?"']
\end{tikzcd}$$

and

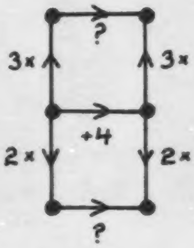
$$\begin{tikzcd}
\bullet \arrow[r, "2\times"] & \bullet \\
\bullet \arrow[u, "6\times"] \arrow[r, "2\times"'] & \bullet \arrow[u, "?"']
\end{tikzcd}$$

(Worksheets N56, N57, N58, N59)

Section 6.2 : Using composition of addition and multiplication functions, label arrows in pictures such as :



and



With the support of a hand-calculator, label arrows which complete a cycle of addition (subtraction), multiplication, and division arrows.

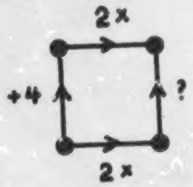
(Worksheets N60, N61, N62)

6.1 Composition #1

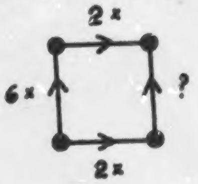
CAPSULE LESSON SUMMARY

Label arrows which complete a cycle of addition and subtraction arrows or of multiplication and division arrows. Use the hand-calculator as support for the situation. Using composition of addition and multiplication functions, label arrows in pictures

such as



and



MATERIALS

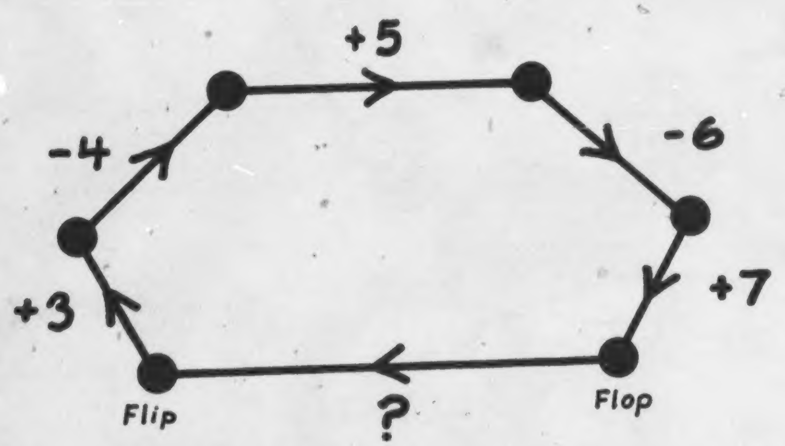
Teacher: Colored chalk

Student: Hand-calculator; Worksheets N56, N57, N58, N59

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board and provide a hand-calculator for each student.



T: Choose any number you wish for Flip and display the number on your calculator. Hide the display and then push the appropriate keys to follow the arrow road until you come to Flop. Next come back to Flip with just one operation. (Trace the arrow labeled "?"). What could this arrow be for?

N 14

Let the students have a few minutes to experiment and to solve this problem.

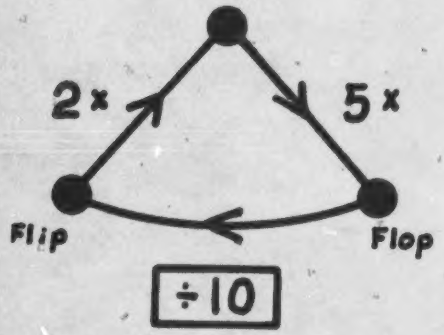
S: -5.

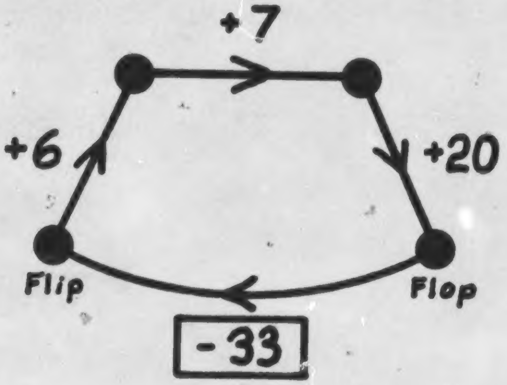
T: How do you know?

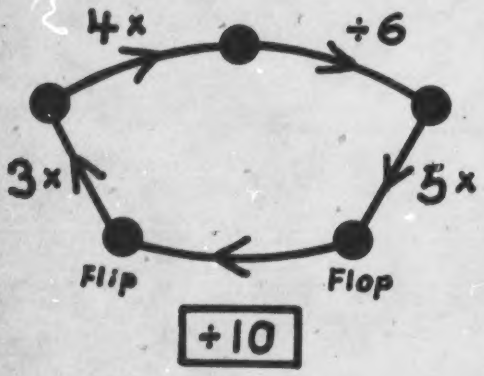
S: I followed the arrow road; +3, -4, +5, -6, +7. I knew that Flop was 5 more than Flip, so the arrow from Flop to Flip could be for -5.

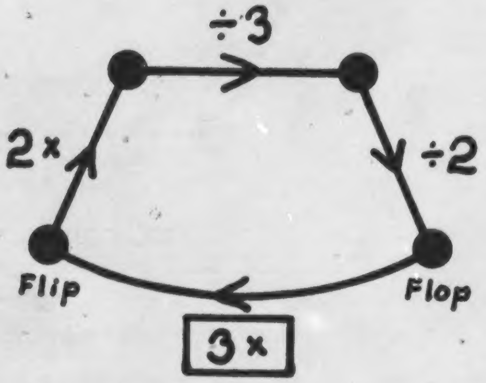
T: Check the display to see if you are correct.

Repeat this activity using some of the arrow roads shown below or choose problems more appropriate to the numerical level of your class. Correct answers are in the boxes.


2×
5×
Flip
Flop
÷10


+7
+6
+20
Flip
Flop
-33


4×
÷6
3×
5×
Flip
Flop
+10


÷3
2×
÷2
Flip
Flop
3×

For the last suggested arrow road of the previous illustration, it is possible that some students will not get Flip even if they correctly press $\boxed{x}\ \boxed{3}\ \boxed{=}$. If this situation occurs, it should be discussed collectively. For example,

T: Let Flip be 5. If we press $\boxed{x}\ \boxed{2}\ \boxed{=}$, which number is on the display?

S: 10.

T: And if we press $\boxed{\div}\ \boxed{3}\ \boxed{=}$?

S: 3.3333333.

T: Is it correct?

S: No, $10 \div 3 = 3\frac{1}{3} = 3.\overset{*}{3}$.

S: The 3's must go on and on. The calculator can only show eight places.

T: Next, we press $\boxed{\div}\ \boxed{2}\ \boxed{=}$. Which number is on the display?

S: 1.6666666.

T: Is it correct?

S: No, $3\frac{1}{3} \div 2 = 10 \div 6 = 1\frac{2}{3} = 1.\overset{*}{6}$.

S: The 6's go on and on, but the calculator only shows the first seven of them.

T: So Flop is 1.666 . . . . Now press $\boxed{x}\ \boxed{3}\ \boxed{=}$. Which number is on the display?

S: 4.9999998.

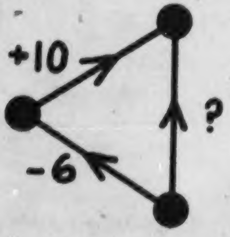
S: But $3 \times 1\frac{2}{3} = 5$.

T: Yes. The calculator has made a very small error in the calculations because it could not show that Flop is $1.\overset{*}{6}$; that is, the 6's go on and on. We should keep this in mind whenever we use a calculator.

It should be mentioned either by a student or by yourself that this difficulty will not occur if a multiple of 3 is chosen for Flip.

Exercise 2

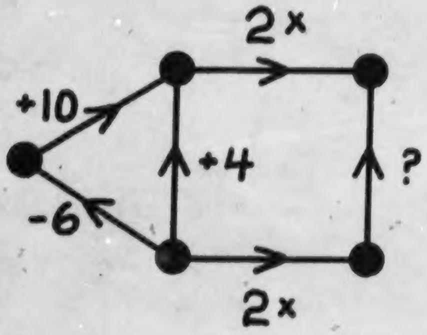
Draw this picture on the board.



T (tracing the arrow labeled "?"): What could this red arrow be for?

S: +4.

Label the arrow "+4" and extend your picture.

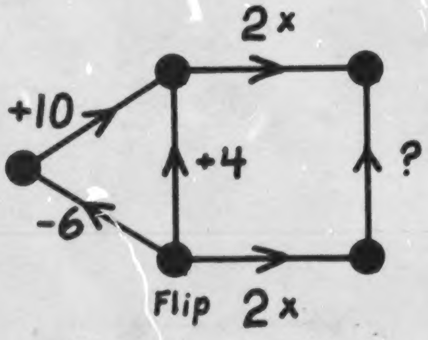


T: What could this red arrow be for?

Let the students have a moment to consider this situation.

S: +8.

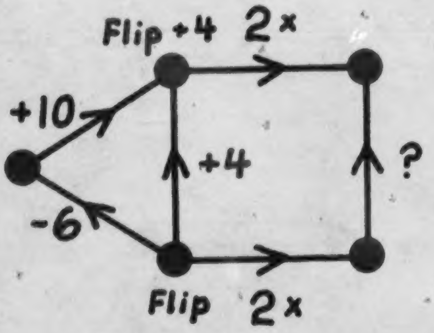
For the benefit of students who have difficulty, label a dot "Flip" as shown in the illustration below.



T (pointing to "Flip"): Suppose there is some number here. Let's call it "Flip". (Trace the +4 arrow to its ending point.) What can we say about the number here?

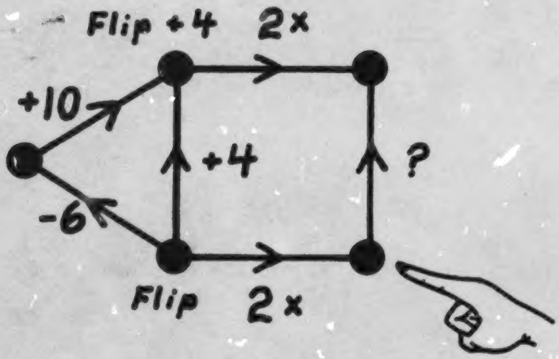
S: It is four more than Flip.

Label the dot "Flip + 4" as shown below.



Point to the dot indicated in the next illustration.

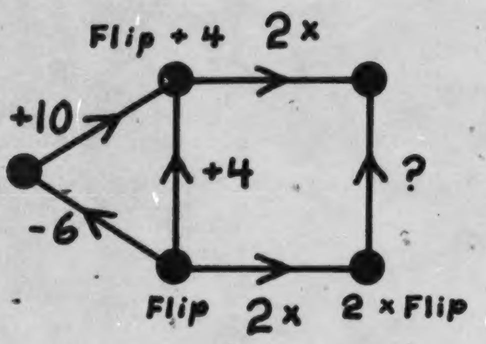
N 14


Flip + 4
2 ×
+10
+4
?
-6
Flip
2 ×

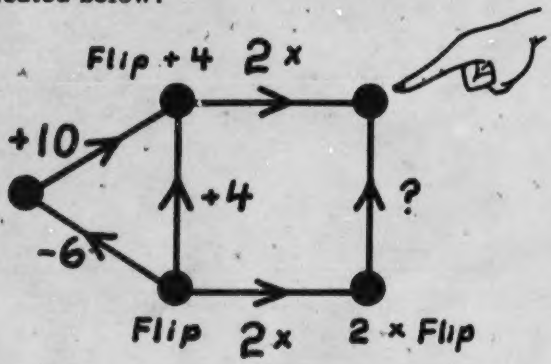
T: What do we know about this number?

S: It is 2 times Flip.

Label the dot accordingly.

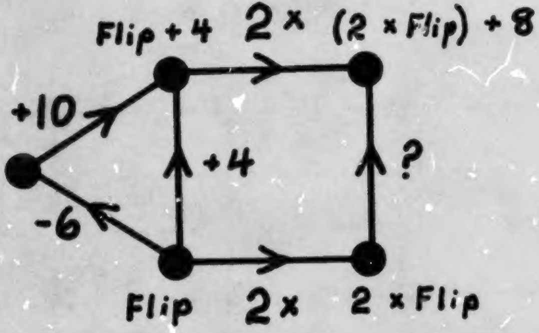

Flip + 4
2 ×
+10
+4
?
-6
Flip
2 ×
2 × Flip

Point to the dot indicated below.


Flip + 4
2 ×
+10
+4
?
-6
Flip
2 ×
2 × Flip

T: What about this number?

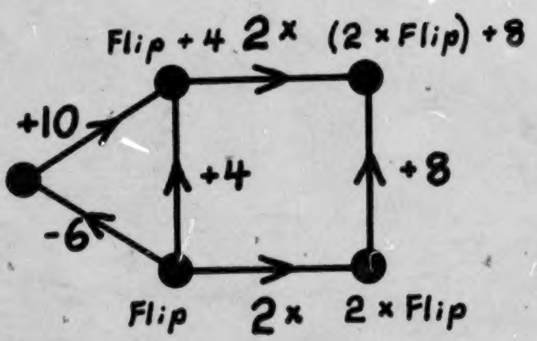
S: It is 2 × (Flip + 4) or (Flip + 4) + (Flip + 4) or (2 × Flip) + 8.


Flip + 4
2×
(2 × Flip) + 8
+10
+4
?
-6
Flip
2×
2 × Flip

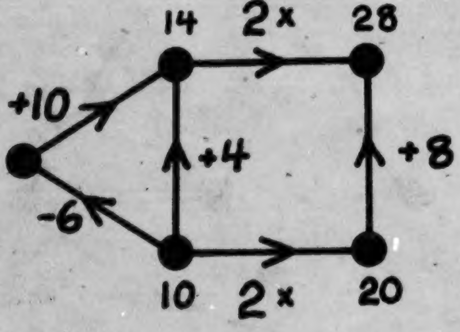
T: What could the red arrow be for?

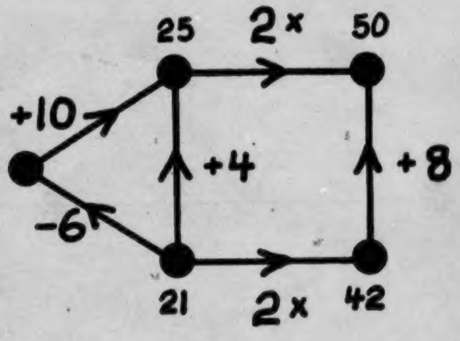
S: +8.

Label the arrow "+8".

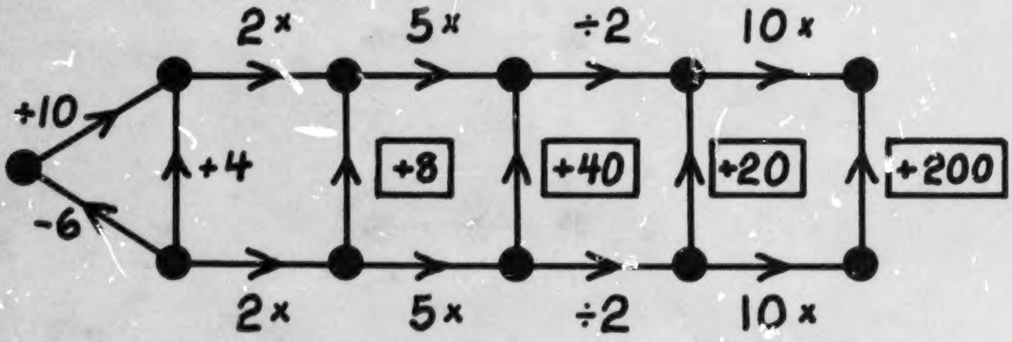

Flip + 4
2×
(2 × Flip) + 8
+10
+4
+8
-6
Flip
2×
2 × Flip

Illustrate the "+8" relationship by choosing several different numbers for Flip and labeling the other dots. For example,


14
2×
28
+10
+4
+8
-6
10
2×
20

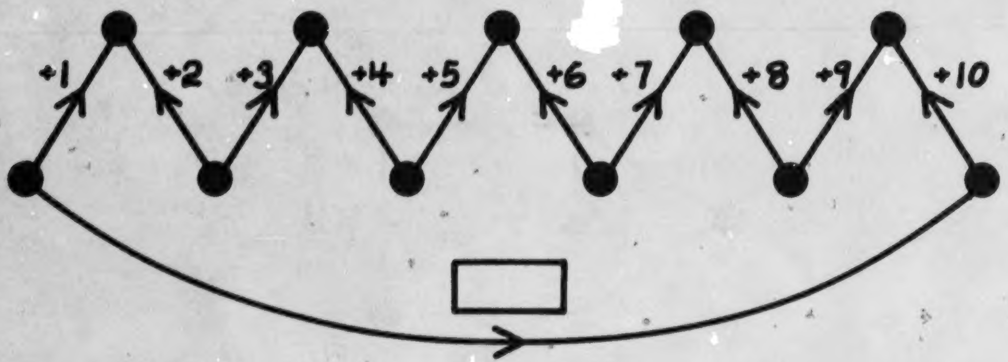

25
2×
50
+10
+4
+8
-6
21
2×
42

Continue this activity by extending the arrow picture and asking students to provide the labels shown in the boxes below.

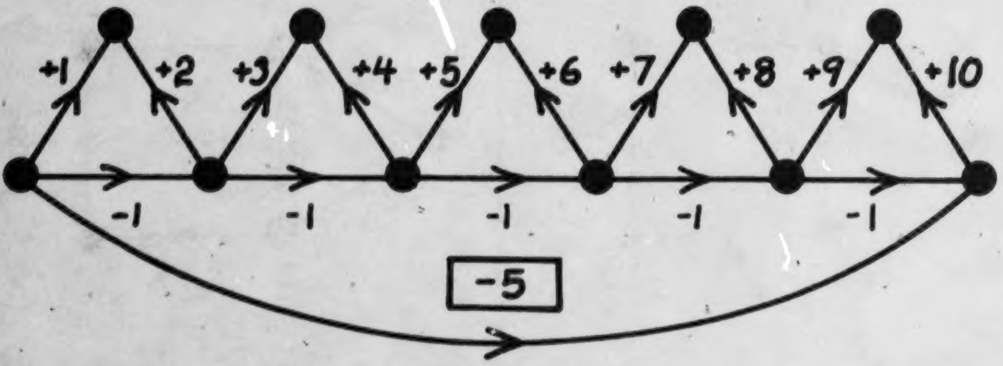

2×
5×
÷2
10×
+10
-6
+4
+8
+40
+20
+200
2×
5×
÷2
10×

Exercise 3

Draw this arrow picture on the board and ask the students to fill in the box.


+1
+2
+3
+4
+5
+6
+7
+8
+9
+10

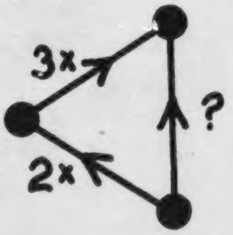
A good clue to give students who have difficulty would be to add the supplementary blue arrows shown in the illustration below. The students will probably notice that each of these could be a - 1 arrow.


+1
+2
+3
+4
+5
+6
+7
+8
+9
+10
-1
-1
-1
-1
-1
-5

Depending on the ability of your class as a whole, you may prefer to end the collective part of the lesson here. With an average or above average class, you should be able to continue with Exercise 4.

Exercise 4

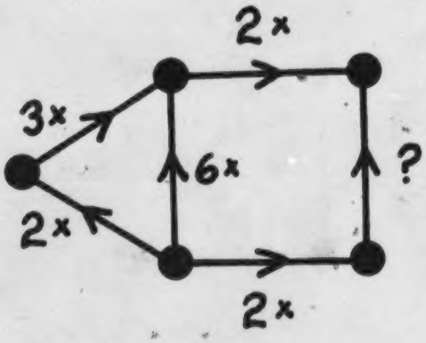
Draw this picture on the board.



T: What could this red arrow be for?

S: 6 x.

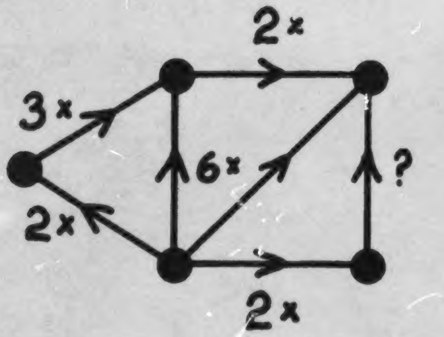
Label the arrow "6 x" and extend your picture.



T: What could this red arrow be for?

Let the students have a moment to consider this situation.

For students who have difficulty, add a supplementary arrow as shown in the next illustration to your picture.



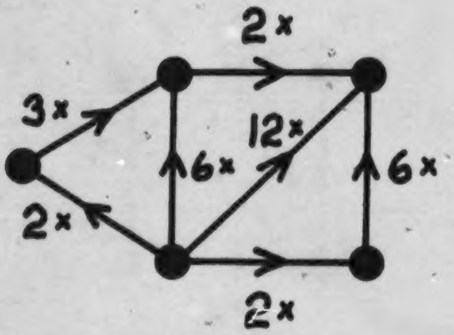
T (tracing the blue arrow, then the 6 x arrow followed by a 2 x arrow): This blue arrow is 6 x followed by 2 x. What could this blue arrow be for?

S: 12 x.

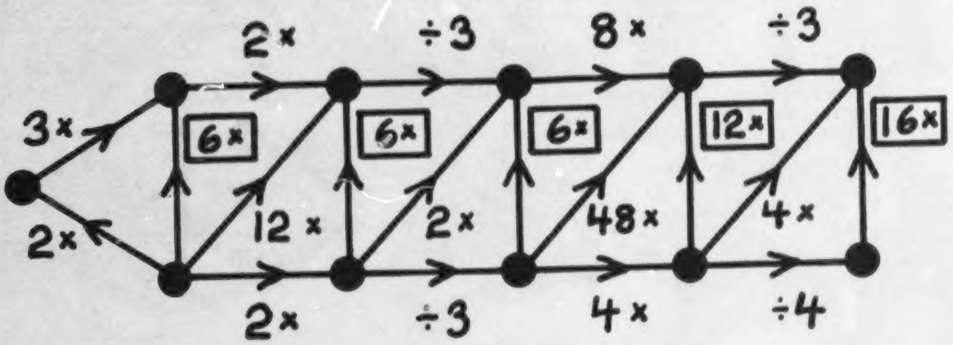
T (tracing the 12 x arrow (blue arrow); then a 2 x arrow followed by the "?" arrow): The 12 x arrow is 2 x followed by which arrow?

S: 6 x.

Label the arrow "6 x".



Continue this activity by extending the arrow picture, each time adding three arrows to your picture and asking students to provide the labels shown in the boxes. Supplementary arrows have been included for your use when appropriate.

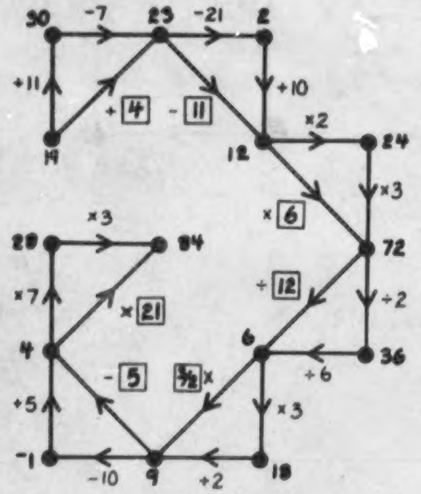

2×
÷3
8×
÷3
3×
2×
6×
6×
6×
12×
16×
12×
2×
48×
4×
2×
÷3
4×
÷4

Worksheets N56, N57, N58, and N59 are available.

Name

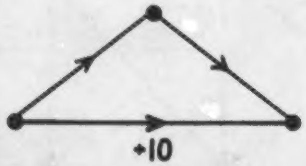
N 56 *

Label the dots and fill in the boxes for the arrows.


30
-7
23
-21
2
+11
+10
+ 4
- 11
14
12
×2
24
×3
×6
72
÷12
÷2
×3
28
84
×7
×21
4
6
36
-5
½ ×
÷6
+5
×3
-1
-10
9
+2
18

Name

N 57 **



Complete the charts.

→	→
+3	+7
+4	+6
-11	+21
-3	+13
+25	-15
-100	+110
- 5	+15
+50	-40

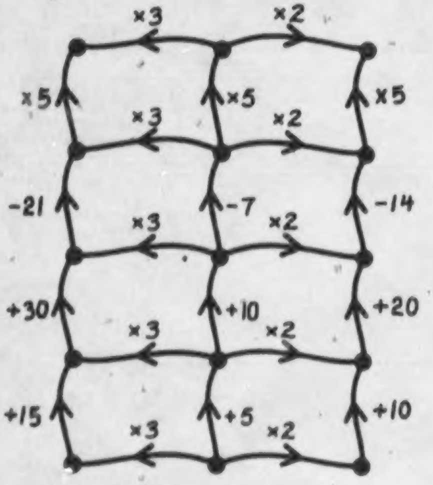
→	→
+3.6	+6.4
+8.1	+1.9
-2.5	+12.5
-3.7	+13.7
+2.65	+7.35
+11.25	-1.25
+0.05	+9.95
-7.43	+17.43

(Many solutions are possible.)

Name

N 58 ***

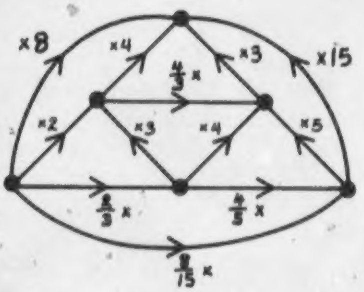
Label the arrows that are not labeled. Each arrow should be labeled +, ×, −, or ÷ some whole number.

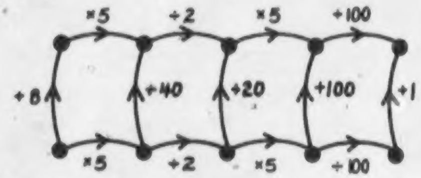

×3
×2
×5
×5
×5
×3
×2
-21
-7
-14
×3
×2
+30
+10
+20
×3
×2
+15
+5
+10
×3
×2

Name

N 59 ****

Label the arrows that are not labeled. Each arrow should be labeled +, ×, −, or ÷ some number.

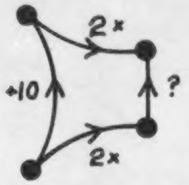

×8
×4
×3
×15
4/3 ×
×2
×3
×4
×5
2/3 ×
4/5 ×
8/15 ×


×5
÷2
×5
÷100
+8
+40
+20
+100
+1
×5
÷2
×5
÷100

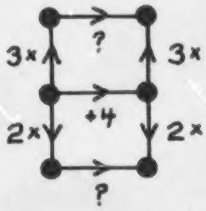
6.2 Composition #2

CAPSULE LESSON SUMMARY

Using composition of addition and multiplication functions, label arrows in pictures such as



and



With the support of a hand-calculator, label arrows which complete a cycle of addition, subtraction, multiplication, and division arrows.

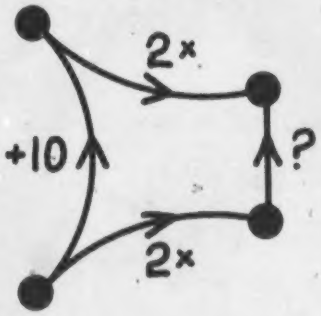
MATERIALS

Teacher: Colored chalk
Student : Hand-calculator; Worksheets N60, N61, N62

DESCRIPTION OF LESSON

Exercise 1

Draw this picture on the board.

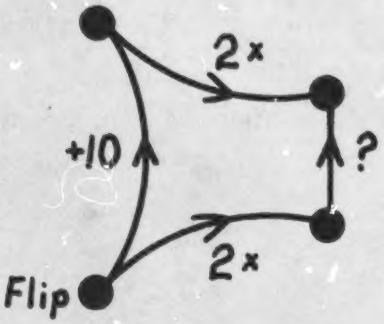


T: What could this red arrow be for?

Let the students have a few minutes to study this situation.

N 29

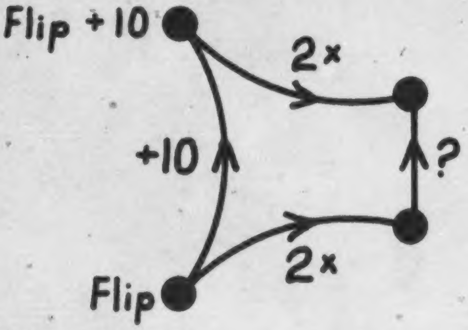
For the benefit of students who have difficulty, label the lower left dot "Flip".


2x
+10
?
2x
Flip

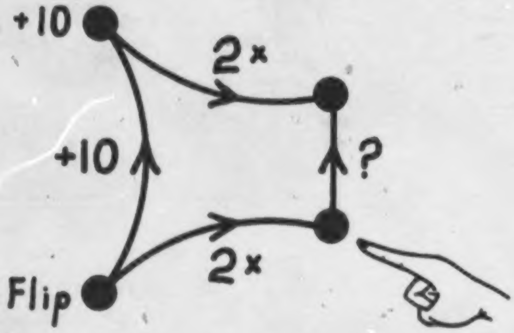
T (pointing to "Flip"): Suppose Flip is here. (Trace the + 10 arrow to its ending point.) What can we say about the number here?

S: It is 10 more than Flip.

Label the dot "Flip + 10" as shown below.


Flip + 10
2x
+10
?
2x
Flip

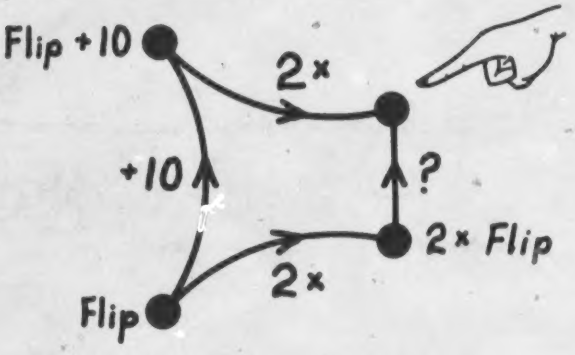
Point to the dot indicated in the illustration below.


Flip + 10
2x
+10
?
2x
Flip

T: What do we know about this number?

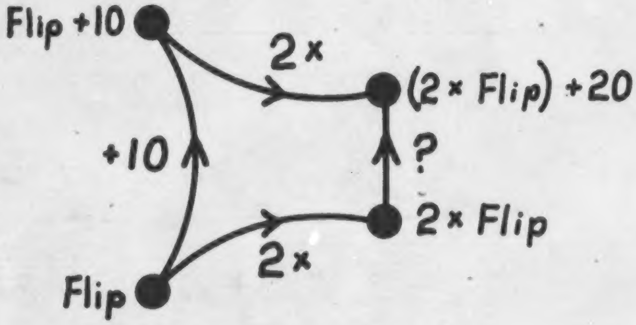
S: It is 2 times Flip.

Label the dot "2 × Flip" and point to the remaining unlabeled dot.


Flip +10
2×
+10
?
2× Flip
2×
Flip

T: What about this number?

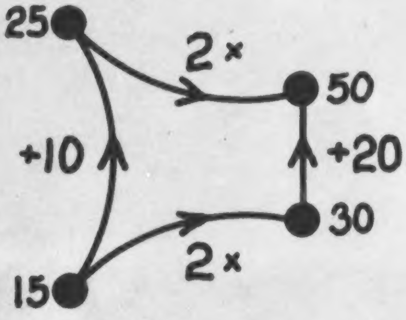
S: It is 2 × (Flip + 10) or (2 × Flip) + 20.

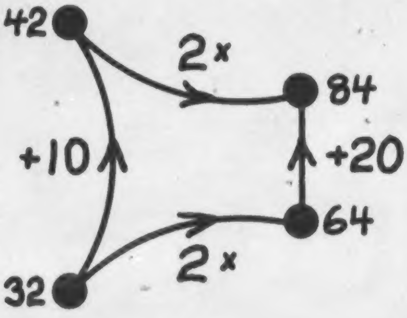

Flip +10
2×
(2× Flip) +20
+10
?
2× Flip
2×
Flip

T (pointing to "?"): What could this arrow be for?

S: +20.

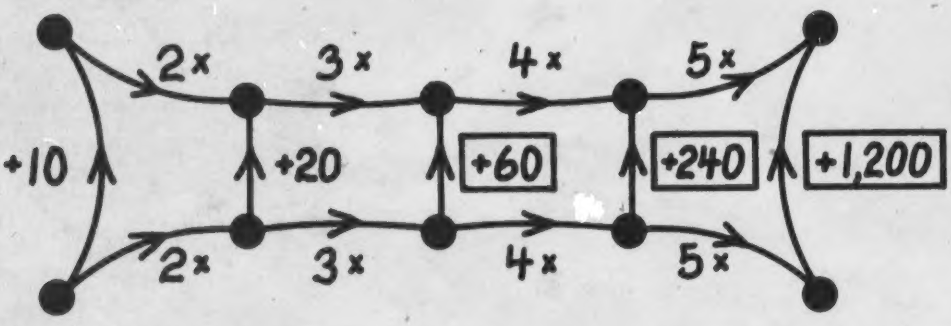
Illustrate the +20 relationship by choosing several different numbers for Flip and then labeling the other dots. For example,


25
2×
50
+10
+20
30
2×
15

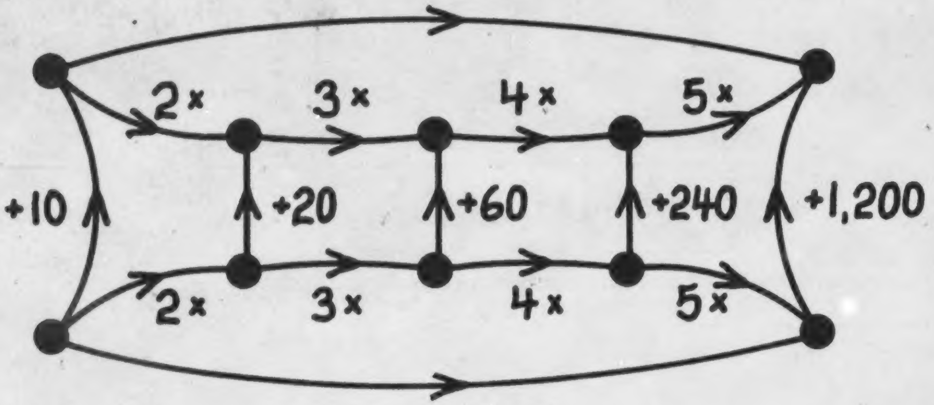

42
2×
84
+10
+20
64
2×
32

N 29

Continue this activity by extending the picture, each time adding three arrows, and asking students to provide the labels shown in boxes below.


2×
3×
4×
5×
+10
+20
+60
+240
+1,200
2×
3×
4×
5×

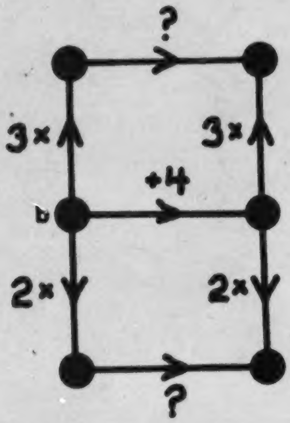
When you have completed this part of the picture, add the two blue arrows shown in the next illustration and ask the students to label them.


2×
3×
4×
5×
+10
+20
+60
+240
+1,200
2×
3×
4×
5×

S: Both blue arrows are for 120 ×; 2 × followed by 3 × followed by 4 × followed by 5 ×.

Exercise 2

Draw this arrow picture on the board.



Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

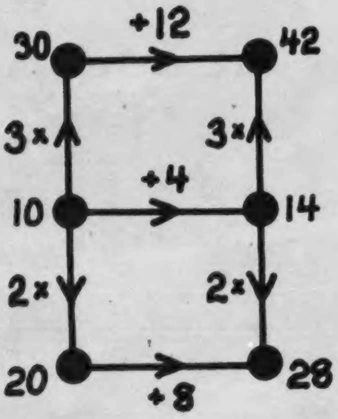
T: How could we label the top and bottom arrows?

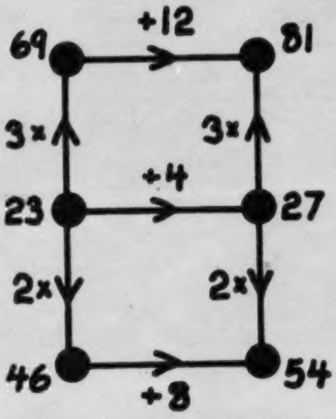
Let the students have a few minutes to consider this situation.

S: The top arrow is for +12.

S: The bottom arrow is for +8.

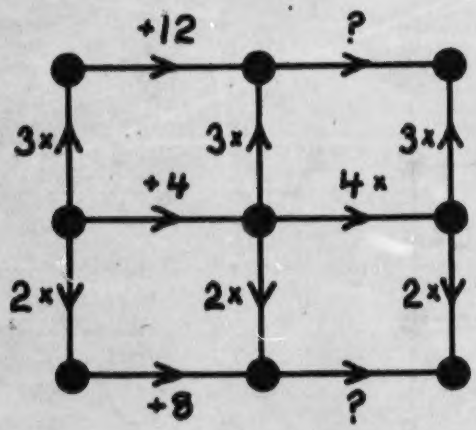
Illustrate the +12 and +8 relationships by choosing several numbers for b and labeling the remaining dots. For example,


30
+12
42
3×
3×
+4
10
14
2×
2×
20
+8
28


69
+12
81
3×
3×
+4
23
27
2×
2×
46
+8
54

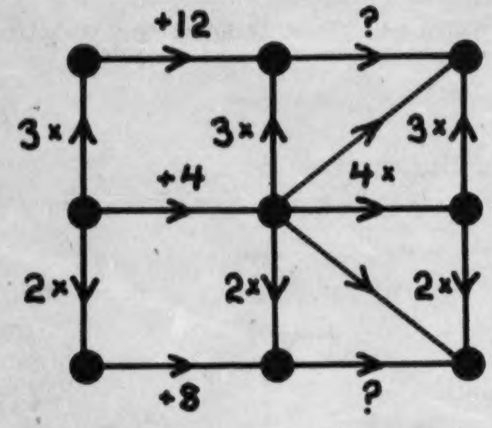
N 29

Extend your arrow picture as illustrated below.



T: What could the top and bottom arrows be for?

A good hint for students who have difficulty labeling the arrows would be to draw supplementary arrows, as shown below, in your picture.



T (tracing the upper blue arrow and then the 4 x arrow followed by the 3 x arrow): This blue arrow is 4 x followed by 3 x. What could this blue arrow be for?

S: 12 x.

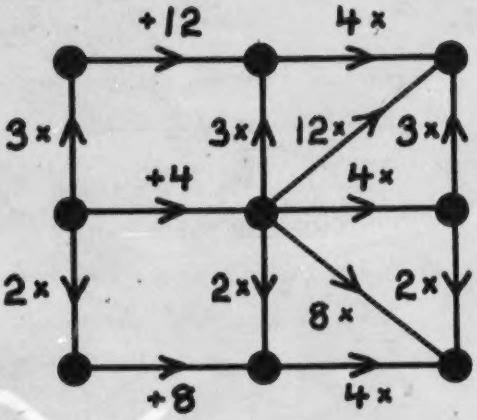
Label the upper blue arrow "12 x".

T (tracing the 12 x arrow and then the 3 x arrow followed by the " ? " arrow):

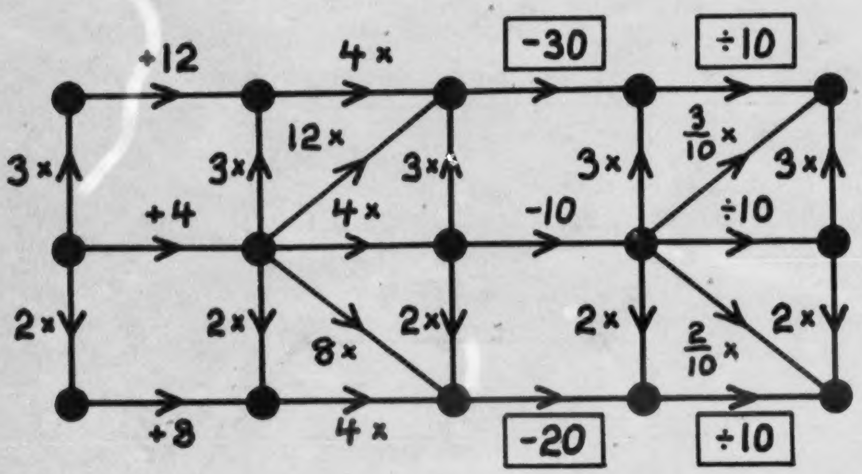
The 12 x arrow is 3 x followed by what?

S: 4 x.

Use a similar line of questioning to find that the lower blue arrow could be an 8 x arrow and then that the lower " ? " arrow could be for 4 x also.

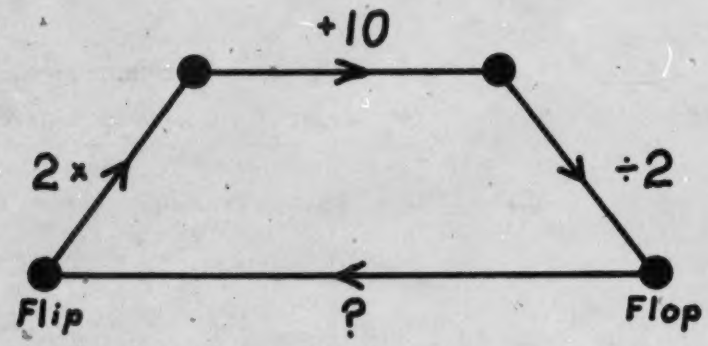

+12
4 x
3 x
3 x
12 x
3 x
+4
4 x
2 x
2 x
8 x
2 x
+8
4 x

Continue this activity by extending the picture; each time adding five arrows to your picture, and asking students to provide the boxed labels. Supplementary arrows have been included for your convenience where appropriate.


+12
4 x
-30
÷10
3 x
3 x
12 x
3 x
3 x
3/10 x
3 x
+4
4 x
-10
÷10
2 x
2 x
8 x
2 x
2 x
2/10 x
2 x
+8
4 x
-20
÷10

Exercise 3

Provide a hand-calculator for each student and draw this arrow picture on the board.



T: Choose any number you wish for Flip and display it on your calculator. Hide the display and then push the appropriate keys to follow the arrow road until you come to Flop. Next come back to Flip with just one operation. (Trace the arrow labeled " ? ".) What could this arrow be for?

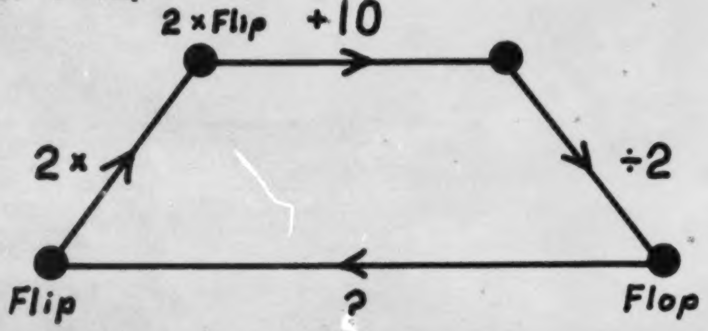
Let the students have a few minutes to explore this situation and then disucss it collectively.

S: -5.

T (pointing to the ending point of the 2 × arrow): What can you tell me about this number?

S: It is 2 times Flip.

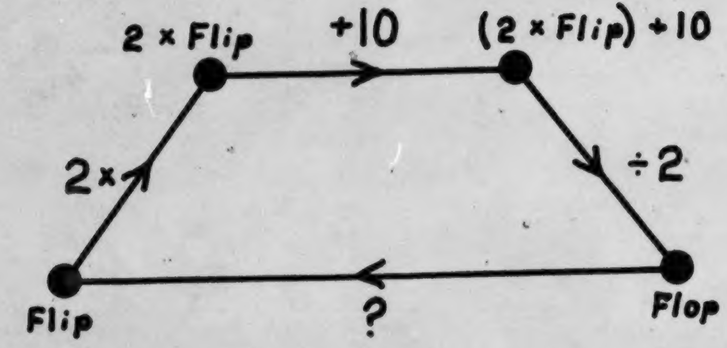
Label the dot " 2 × Flip".



T (pointing to the ending point of the +10 arrow): And what about this number?

S: It is (2 × Flip) + 10.

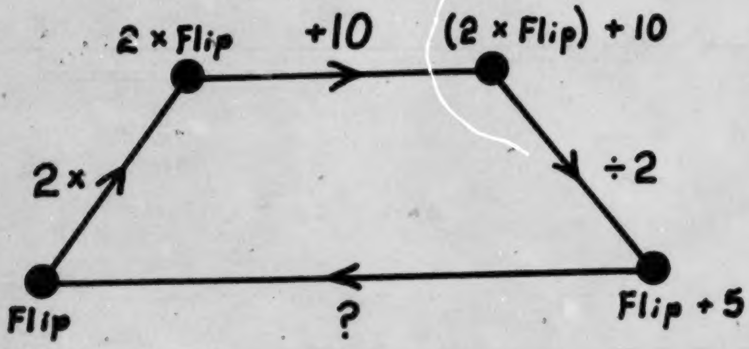
Label the dot "(2 × Flip) + 10".


2 × Flip
+10
(2 × Flip) + 10
2×
÷2
Flip
?
Flop

T (pointing to "Flop"): What can you tell me about this number? Be careful!

S: Flip plus 5.

S: Take $\frac{1}{2}$ of 2 times Flip and $\frac{1}{2}$ of 10.

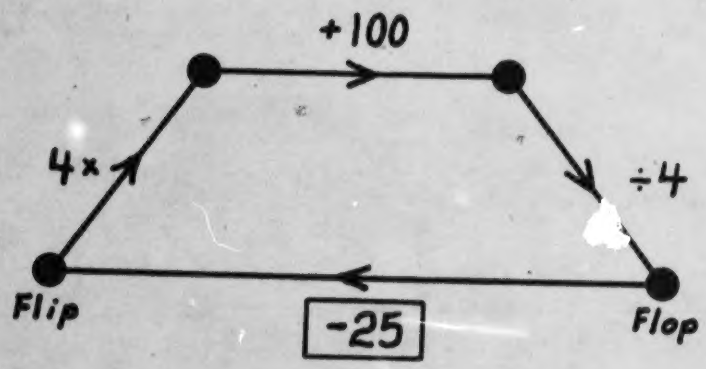

2 × Flip
+10
(2 × Flip) + 10
2×
÷2
Flip
?
Flip + 5

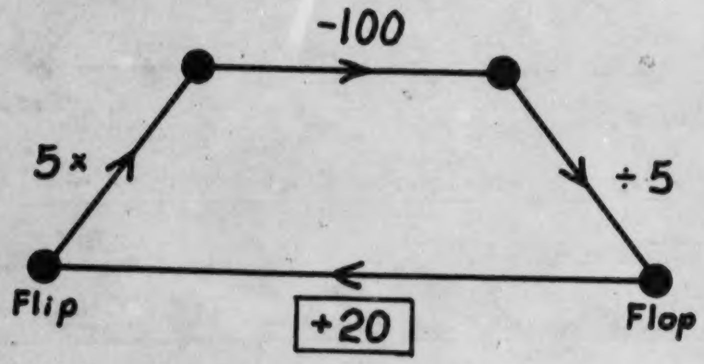
T (tracing the "?" arrow): What could this arrow be for?

S: -5.

Repeat this activity using the two arrow roads in the next illustration. Correct answers are in the boxes.

N 29


+100
4×
÷4
Flip
Flop
-25

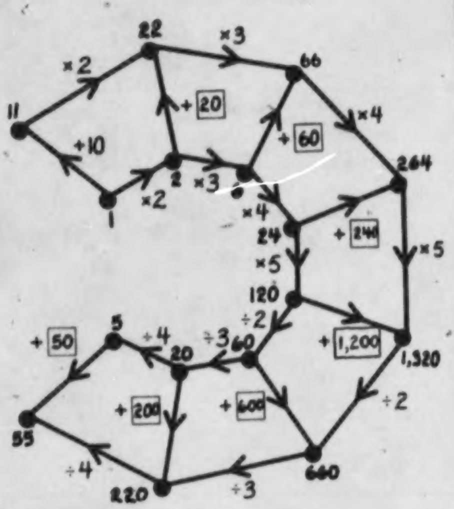

-100
5×
÷5
Flip
Flop
+20

Worksheets N60, N61, and N62 are available.

Name

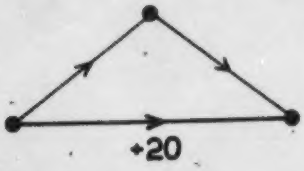
N 60 *

Label the dots and fill in the boxes for the arrows.


22
×3
×2
66
+ 20
11
+10
+ 60
×4
2
×3
6
264
1
×2
×4
24
+ 240
×5
×5
120
÷2
5
÷4
÷3
60
+ 1,200
+ 50
20
1,320
+ 200
+ 600
÷2
55
÷4
220
÷3
660

Name

N 61 **



Complete the charts.

→	→
+13	+7
+25	-5
-16	+36
-8	+28
+32	-12
-105	+125
-55	+75

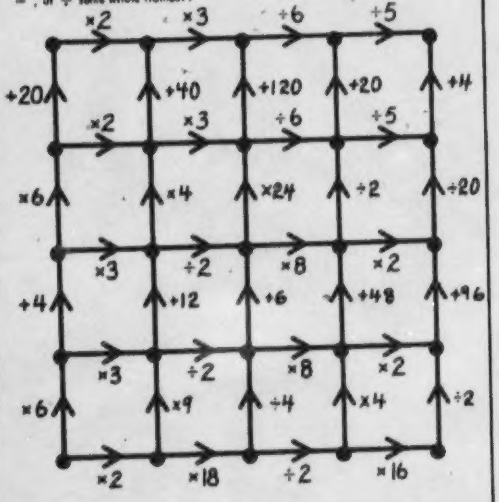
→	→
+7.5	+12.5
+7.2	+12.8
-2.4	+22.4
-10.5	+30.5
+8.75	+11.25
+23.25	-3.25
+1,001	-981

(Many solutions are possible.)

Name

N 62 ***

Label each arrow that is not labeled. Each arrow should be labeled +, ×, −, or ÷ some whole number.


×2
×3
÷6
÷5
+20
+40
+120
+20
+4
×2
×3
÷6
÷5
×6
×4
×24
÷2
÷20
×3
÷2
×8
×2
+4
+12
+6
+48
+96
×3
÷2
×8
×2
×6
×9
÷4
×4
÷2
×2
×18
÷2
×16

7

HAND-CALCULATOR PUZZLES

In this chapter, the lessons exploit the hand-calculator to present situations involving mental arithmetic. Basically, students explore various ways to display a given number on a "special" hand-calculator, that is, a hand-calculator with only the keys $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{9}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$. An additional challenge is to find a solution pressing as few keys as possible. Using these "special" hand-calculators, students determine ways to multiply and to divide by 2 or 3 or 4 or 10. These latter situations involve observing, for example, that $\frac{1}{3} = \frac{3}{9}$ and $\frac{1}{10} = \frac{3}{30}$. From such observations, one finds

$$\div 3 = (\boxed{+}\boxed{=}\boxed{=}) \circ \div 9$$

$$\div 10 = (\boxed{+}\boxed{=}\boxed{=}) \circ \div 5 \circ \div 6$$

NOTE: $\boxed{+}\boxed{=}\boxed{=}$ is a way to multiply by 3 with the "special" hand-calculator.

CAPSULE LESSON SUMMARIES

Section 7.1: Put various numbers on the display of a hand-calculator using only the keys $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{9}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$. Find solutions which require pressing keys less than ten times.

Section 7.2: Put various numbers on the display of a "special" hand-calculator with only the keys $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{9}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$. Find solutions which require pressing keys less than ten times. Explore ways to use the special calculator to double a number, divide a number by 2, calculate 10 × a number, and divide a number by 10.

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Section 7.3: Put various numbers on the display of a "special" hand-calculator with only the keys $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{9}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$. Find solutions which require pressing keys less than ten times. Explore ways to use the special calculator to multiply a number by 3 or 4 and to divide a number by 3 or 4.

7.1 Hand-Calculator Puzzles #1

CAPSULE LESSON SUMMARY

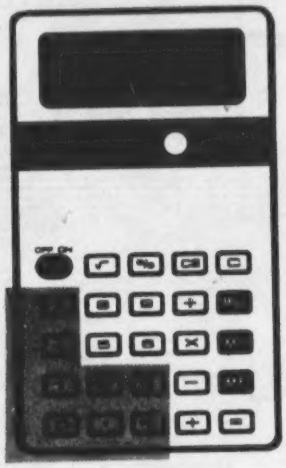
Put various numbers on the display of a hand-calculator using only the keys $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{9}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, [illegible], and $\boxed{=}$. Find solutions which require pressing keys less than ten times.

MATERIALS

Teacher: Hand-calculator
Student: Hand-calculator; paper; pencil

DESCRIPTION OF LESSON

NOTE: Prior to the presentation of this lesson, cover with masking tape (or any other suitable tape) the keys, shown in the illustration below, of each hand-calculator[*] that will be used during this lesson.



Distribute one of these "special" hand-calculators to each student.

---

[*] For the lesson descriptions in this chapter, we assume that the calculators being used have the following characteristic: if you start with some number on the display and press $\boxed{+}$ and then $\boxed{=}$ n times, the number on the display is n + 1 times your number. See Section 2 (f) of the General Introduction for additional comments on such features of hand-calculators.

N 18

Exercise 1

T: You may use only the keys that are not covered by tape. Try to put 40 on the display of your calculator.

Let the students have a few minutes to explore this very open situation. Walk around the room observing the students' work and encouraging those who need it.

There are many solutions to this problem. Since we cannot predict which solutions your students will find, we will present some of the solutions discovered by students in our development classes.

S: Press $\boxed{5}\boxed{\times}\boxed{8}\boxed{=}$.

Write the students solution on the board.

$$40: \boxed{5}\ \boxed{\times}\ \boxed{8}\ \boxed{=}$$

T: Are there other solutions?

S: Press $\boxed{9}\boxed{9}\boxed{-}\boxed{5}\boxed{9}\boxed{=}$.

Add this solution to your list on the board.

$$40: \boxed{5}\ \boxed{\times}\ \boxed{8}\ \boxed{=}$$

$$\boxed{9}\ \boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{9}\ \boxed{=}$$

Accept several more solutions before continuing with other numbers. Sometimes ask the class to check a solution using their calculators. It is not necessary to verify collectively every solution. Some other solutions are presented below.*

40: $\boxed{9}\ \boxed{\times}\ \boxed{5}\ \boxed{-}\ \boxed{5}\ \boxed{=}$

$\boxed{8}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}$

$\boxed{5}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}$

$\boxed{9}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{+}\ \boxed{8}\ \boxed{+}\ \boxed{5}\ \boxed{=}$

Repeat this activity with 400. A few solutions for putting 400 on the display are given below.

400: $\boxed{9}\ \boxed{9}\ \boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{9}\ \boxed{9}\ \boxed{=}$

$\boxed{8}\ \boxed{8}\ \boxed{-}\ \boxed{8}\ \boxed{\times}\ \boxed{5}\ \boxed{=}$

$\boxed{9}\ \boxed{6}\ \boxed{\times}\ \boxed{5}\ \boxed{-}\ \boxed{8}\ \boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{=}$

$\boxed{5}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{\times}\ \boxed{=}$

This activity offers an excellent opportunity for students to explore their hand-calculators in a very open and creative way. Some students will find solutions that make use of interesting characteristics of the calculators. In such cases, these solutions should be discussed collectively.

---

* We assume for the three lessons in this chapter that calculators with an automatic constant feature are used. For a more detailed discussion of hand-calculators, see the General Introduction Section 2 (f) on Role and Use of Hand-Calculators.

For example, consider the last solution given for 400.

T: Press $\boxed{5}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ and watch the display.

S: The calculator adds 5 each time. 20 is on the display now.

T: Now press $\boxed{\times}\ \boxed{=}$. Which number is on the display? Why?

S: 400. The calculator multiplies 20 by 20 and 20 × 20 = 400.

Exercise 2

T: Try to display 7. Each time you press a key it costs one cent. Try to find as cheap a solution as you can. Try to spend less than ten cents.

Let the students have a few minutes to work independently. When it is obvious that many students have at least one solution costing less than ten cents, write a solution on the board. For example,

$$7:\quad \boxed{8}\ \boxed{+}\ \boxed{5}\ \boxed{-}\ \boxed{6}\ \boxed{=}$$

T: Here is a six cent solution. Try to find some other solutions (cheaper, if possible).

Encourage the students to find several solutions using the different operation keys. The following solutions with their respective prices were found by students in our development classes.

7: $\boxed{6}\ \boxed{6}\ \boxed{-}\ \boxed{5}\ \boxed{9}\ \boxed{=}$ 6¢

$\boxed{5}\ \boxed{6}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$ 5¢

$\boxed{9}\ \boxed{\times}\ \boxed{8}\ \boxed{-}\ \boxed{6}\ \boxed{5}\ \boxed{=}$ 7¢

Continue this activity with other numbers such as 54, 57, 64, 125, and 404. Students should be encouraged to find solutions using as many different operations as possible. Several solutions to each problem are provided here.

54: $\boxed{5}\ \boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{=}$ 5¢

$\boxed{6}\ \boxed{\times}\ \boxed{9}\ \boxed{=}$ 4¢

$\boxed{9}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ 7¢

57: $\boxed{6}\ \boxed{6}\ \boxed{-}\ \boxed{9}\ \boxed{=}$ 5¢

$\boxed{5}\ \boxed{\times}\ \boxed{9}\ \boxed{+}\ \boxed{6}\ \boxed{=}\ \boxed{=}$ 7¢

$\boxed{5}\ \boxed{6}\ \boxed{\div}\ \boxed{8}\ \boxed{\times}\ \boxed{=}\ \boxed{+}\ \boxed{8}\ \boxed{=}$ 9¢

N 18

64:	$\boxed{8}\ \boxed{\times}\ \boxed{8}\ \boxed{=}$	4¢
	$\boxed{8}\ \boxed{\times}\ \boxed{=}$	3¢
	$\boxed{5}\ \boxed{9}\ \boxed{+}\ \boxed{5}\ \boxed{=}$	5¢
	$\boxed{6}\ \boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{=}$	5¢
125:	$\boxed{6}\ \boxed{9}\ \boxed{+}\ \boxed{5}\ \boxed{6}\ \boxed{=}$	6¢
	$\boxed{5}\ \boxed{\times}\ \boxed{=}\ \boxed{=}$	4¢
404:	$\boxed{9}\ \boxed{5}\ \boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{5}\ \boxed{5}\ \boxed{=}$	8¢
	$\boxed{6}\ \boxed{\times}\ \boxed{6}\ \boxed{6}\ \boxed{+}\ \boxed{8}\ \boxed{=}$	7¢

Some of your students who finish quickly might enjoy these two more challenging problems.

1,111:	$\boxed{9}\ \boxed{9}\ \boxed{9}\ \boxed{9}\ \boxed{\div}\ \boxed{9}\ \boxed{=}$	7¢
	$\boxed{5}\ \boxed{5}\ \boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{5}\ \boxed{6}\ \boxed{=}$	8¢
8,000:	$\boxed{8}\ \boxed{8}\ \boxed{8}\ \boxed{8}\ \boxed{-}\ \boxed{8}\ \boxed{8}\ \boxed{8}\ \boxed{=}$	9¢
	$\boxed{8}\ \boxed{\times}\ \boxed{9}\ \boxed{9}\ \boxed{9}\ \boxed{+}\ \boxed{8}\ \boxed{=}$	8¢

Exercise 3

NOTE: If you class is weaker than average, you may prefer to omit this exercise.

This exercise continues in the same spirit as Exercise 1; the difference being that here we seek ways to display decimals less than 1.

T: Using only the keys which are not covered, try to display 0.5.

Let the students have an opportunity to explore this situation. If some students need help getting started, try the following line of questioning.

T: What are some fractional names for 0.5?

As the students suggest some names, write them on the board.

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \cdots$$

S: We can divide by 6 or by 8.

S: I used "$\frac{3}{6}$". Press $\boxed{8}\ \boxed{-}\ \boxed{5}\ \boxed{\div}\ \boxed{6}\ \boxed{=}$.

S: I used "$\frac{4}{8}$". Press $\boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$.

Continue this activity with other numbers such as 0.1, 0.2, and 0.3. Try to discuss solutions as soon as they are discovered; sometimes this permits other students to build new solutions from the previous ones.

For example, in one of our development classes, some students knew that $0.5 \div 5 = 0.1$ and were able to find the following solution for 0.1.

$$0.1: \boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{\div}\ \boxed{8}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$$

N 18

Some other solutions for 0.1 and solutions for 0.2 and 0.3 are given below.

0.1: $\boxed{8}\ \boxed{-}\ \boxed{5}\ \boxed{\div}\ \boxed{6}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

$\boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{\div}\ \boxed{=}\ \boxed{=}$

$\boxed{6}\ \boxed{5}\ \boxed{-}\ \boxed{5}\ \boxed{5}\ \boxed{\div}\ \boxed{=}\ \boxed{=}$

0.2: $\boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{\div}\ \boxed{=}\ \boxed{=}\ \boxed{+}\ \boxed{=}$

$\boxed{5}\ \boxed{\div}\ \boxed{5}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

$\boxed{5}\ \boxed{\div}\ \boxed{5}\ \boxed{=}\ \boxed{=}$

0.3: $\boxed{9}\ \boxed{\div}\ \boxed{6}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

$\boxed{6}\ \boxed{+}\ \boxed{6}\ \boxed{\div}\ \boxed{8}\ \boxed{\div}\ \boxed{5}\ \boxed{=}$

7.2 Hand-Calculators Puzzles #2

CAPSULE LESSON SUMMARY

Put various numbers on the display of a "special" hand-calculator with only the keys $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{9}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$. Find solutions which require pressing keys less than ten times. Explore ways to use the special calculator to double a number, divide a number by 2, calculate 10× a number, and divide a number by 10.

MATERIALS

Teacher: Hand-calculator; colored chalk
Student: Hand-calculator; paper; pencil

DESCRIPTION OF LESSON

Distribute specially prepared hand-calculators, (see the note at the beginning of Lesson N 7.1) one to each student.

Exercise 1

T: You may use only the keys that are not covered with tape. Put 31 on the display of your calculator. Each time you press a key it costs one cent. Try to spend less than ten cents.

Let the students have a few minutes to explore this vary open situation. Walk around the room observing the students' work and encouraging those who need it.

There are many solutions to this problem. Since we cannot predict which solutions your students will find, we will present some of the solutions discovered by students in our development classes.

S: Press $\boxed{8}\ \boxed{\times}\ \boxed{5}\ \boxed{-}\ \boxed{9}\ \boxed{=}$.

Write the student's solution and the price of the solution on the board.

31: $\boxed{8}\ \boxed{\times}\ \boxed{5}\ \boxed{-}\ \boxed{9}\ \boxed{=}$ 6¢

N 27

T: Try to find other ways to put 31 on the display.

Accept several more solutions before continuing with other numbers. Sometimes ask the class to check a solution using their calculators. It is not necessary to verify collectively every solution. Some other solutions and their prices are presented below.

31: $\boxed{6}\ \boxed{\times}\ \boxed{6}\ \boxed{-}\ \boxed{5}\ \boxed{=}$ 6¢

$\boxed{9}\ \boxed{9}\ \boxed{-}\ \boxed{6}\ \boxed{8}\ \boxed{=}$ 6¢

$\boxed{9}\ \boxed{\times}\ \boxed{5}\ \boxed{-}\ \boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{=}$ 8¢

$\boxed{9}\ \boxed{8}\ \boxed{+}\ \boxed{8}\ \boxed{8}\ \boxed{\div}\ \boxed{6}\ \boxed{=}$ 8¢

$\boxed{9}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{-}\ \boxed{5}\ \boxed{=}$ 8¢

Continue this activity with other numbers such as 111 and 777. Encourage the students to find several solutions using various combinations of the $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, and $\boxed{\div}$ keys. Some of the many solutions to each problem are presented here.

111: $\boxed{9}\ \boxed{9}\ \boxed{+}\ \boxed{6}\ \boxed{=}\ \boxed{=}$ 6¢

$\boxed{9}\ \boxed{9}\ \boxed{9}\ \boxed{\div}\ \boxed{9}\ \boxed{=}$ 6¢

$\boxed{5}\ \boxed{\times}\ \boxed{=}\ \boxed{=}\ \boxed{-}\ \boxed{8}\ \boxed{-}\ \boxed{6}\ \boxed{=}$ 9¢

$\boxed{5}\ \boxed{5}\ \boxed{+}\ \boxed{=}\ \boxed{+}\ \boxed{9}\ \boxed{-}\ \boxed{8}\ \boxed{=}$ 9¢

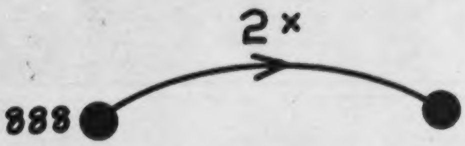
77: $\boxed{6}\ \boxed{9}\ \boxed{+}\ \boxed{8}\ \boxed{=}$ 5¢

$\boxed{8}\ \boxed{\times}\ \boxed{9}\ \boxed{+}\ \boxed{5}\ \boxed{=}$ 6¢

$\boxed{9}\ \boxed{\times}\ \boxed{9}\ \boxed{-}\ \boxed{9}\ \boxed{+}\ \boxed{5}\ \boxed{=}$ 8¢

Exercise 2

Draw this arrow picture on the board.



T: Put some number on the display of your hand-calculator. I have chosen 888, but you may choose any number you wish. Try to find the cheapest way to double your number. Try to find a way that would work for any number we choose.

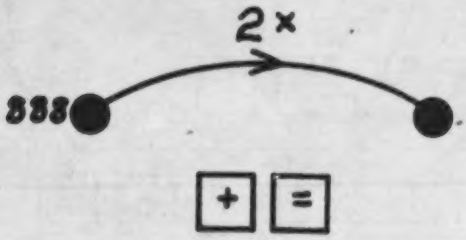
Let the students have a few minutes to work independently.

S: Press $\boxed{+}\,\boxed{=}$ and you can double any number for two cents.

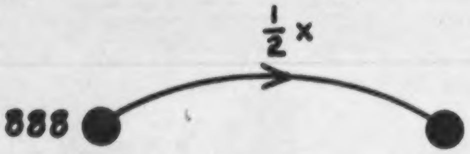
Perhaps other ways of doubling particular numbers will be found, but none will be cheaper than two cents.

N 27

Add this information to your arrow picture.



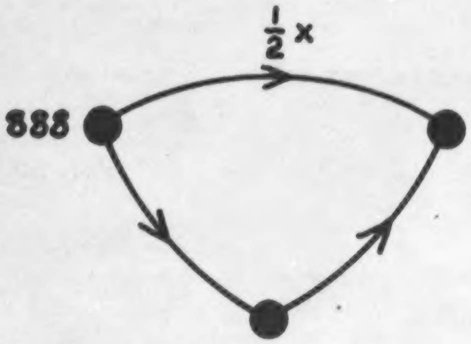
Erase the arrow labels and label the arrow "$\frac{1}{2}$x".



T: How could we halve a number? Try to find the cheapest way. Find a method which works for any number, not just 888 or your number.

Let the students work independently for a few minutes. If there are many students who are having difficulty, suggest they consider a detour.

Add two arrows to your picture.



T: Do you know a detour for $\frac{1}{2}$x; that is, a composition of arrows which is the same as $\frac{1}{2}$x?

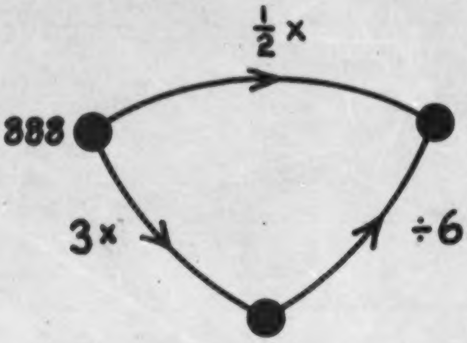
S: 1 × followed by ÷ 2 (or just ÷ 2).

T: That is one way, but we do not have a "1" or a "2" to use. Can we do it another way?

S: We could use 3 × followed by ÷ 6.

If this suggestion is not forthcoming, make it yourself.

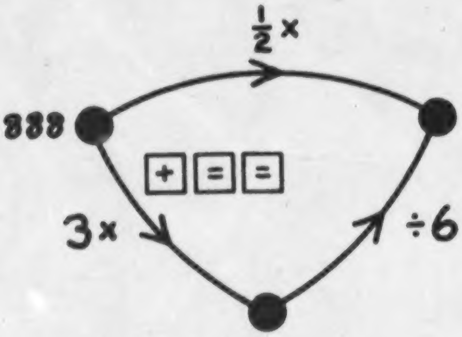
Label the arrows appropriately.



T(tracing the 3 × arrow): Can we triple a number easily using these special hand-calculators?

S: Yes, press $\boxed{+}\,\boxed{=}\,\boxed{=}$.

Add this information to your picture.



N 27

T: We can divide by 6 easily with our special calculators, so we can divide by 2 by following this detour. Use your calculator to calculate "888 ÷ 2".

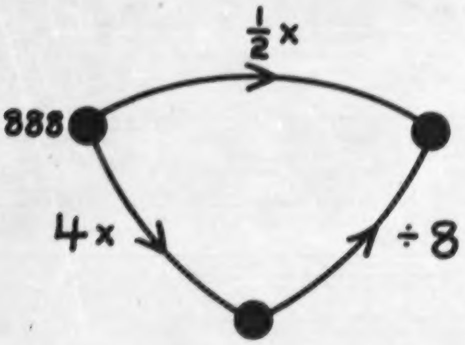
The students should verify that the above method works by putting 888 on the display and pressing $\boxed{+}\,\boxed{=}\,\boxed{=}\,\boxed{\div}\,\boxed{6}\,\boxed{=}$.

Erase the labels from the red arrows.

T: Can you suggest another way to halve a number?

S: 4× followed by ÷ 8.

Add this information to your picture.

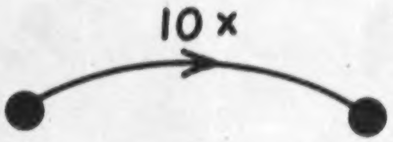


T: How do we perform the calculation on our special calculators?

S: Press $\boxed{+}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{\div}\,\boxed{8}\,\boxed{=}$.

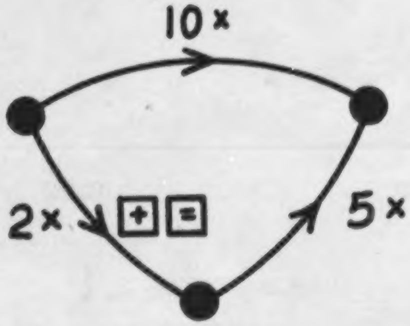
Repeat this exercise for 10× and ÷ 10.

For 10×, some students might suggest:



$\boxed{+}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{=}\,\boxed{=}$

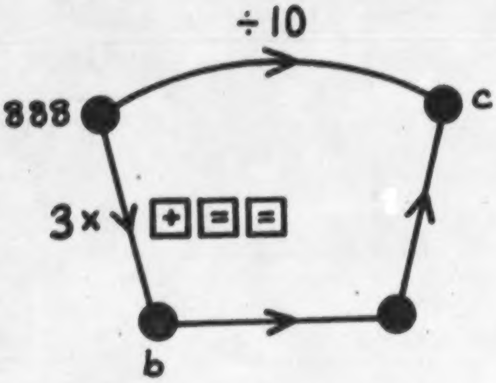
However, a cheaper solution might be found if you provide a detour. A solution is illustrated below.



For ÷10 you might suggest a detour with three arrows.

T: How can we divide a number by 10 (or find $\frac{1}{10}$ of a number) with our special calculators? I have found a solution using a detour with three arrows.

Let the students have a few minutes to work independently. Some students may need some help getting started. In that case, add three arrows to your picture as shown below.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (tracing an imagined arrow starting from b and ending at c): 3× followed by what is the same as ÷10 (or $\frac{1}{10}$×)?

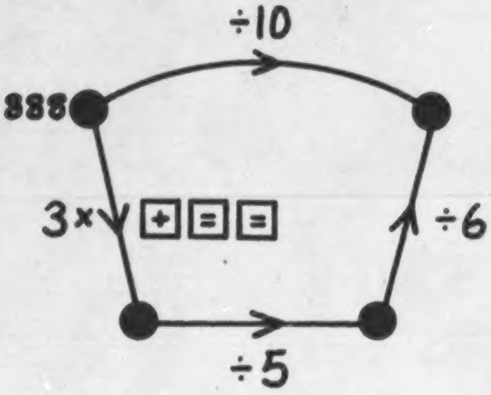
N 27

S: ÷ 30.

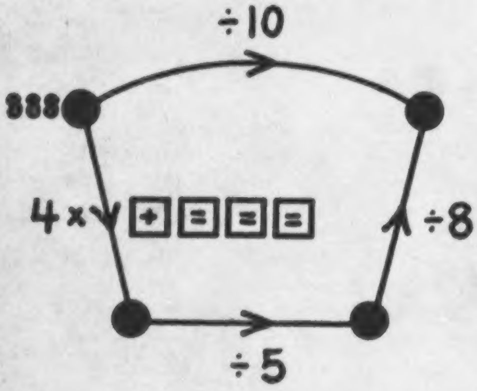
T: Good. How can we divide by 30 with our special calculators?

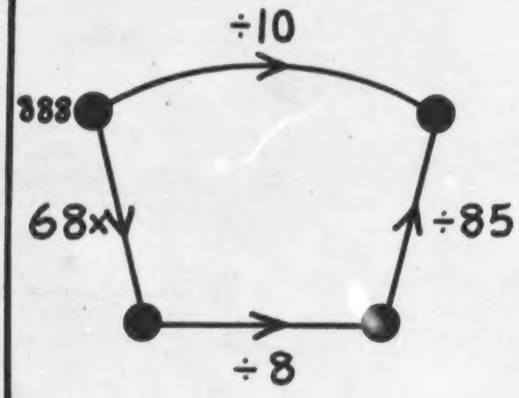
S: ÷ 5 followed by ÷ 6.

Label the arrows accordingly.



If time permits, your students could find other solutions to this problem. Two other solutions are given below.




÷10
888
68×
÷85
÷8

7.3 Hand-Calculator Puzzles #3

CAPSULE LESSON SUMMARY

Put various numbers on the display of a "special" hand-calculator with only the keys $\boxed{5}$, $\boxed{6}$, $\boxed{8}$, $\boxed{9}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$. Find solutions which require pressing keys less than ten times. Explore ways to use the "special" calculator to multiply a number by 3 or 4 and to divide a number by 3 or 4.

MATERIALS

Teacher: Hand-calculator; colored chalk
Student: Hand-calculator; paper; pencil

DESCRIPTION OF LESSON

Distribute specially prepared hand-calculators, (see the note at the beginning of Lesson N 7.1) one to each student.

Exercise 1

T: You may use only the keys that are not covered with tape. Put 33 on the display of your calculator. Each time you press a key it costs one cent. Try to spend less than ten cents.

Let the students have several minutes to work independently while you observe their work. Encourage students to find several solutions.

Several solutions and their prices are presented below. These solutions were suggested by students in our development classes.

33: $\boxed{8}\ \boxed{8}\ \boxed{-}\ \boxed{5}\ \boxed{5}\ \boxed{=}$ 6¢

$\boxed{6}\ \boxed{\times}\ \boxed{5}\ \boxed{-}\ \boxed{5}\ \boxed{+}\ \boxed{8}\ \boxed{=}$ 8¢

$\boxed{5}\ \boxed{5}\ \boxed{-}\ \boxed{8}\ \boxed{-}\ \boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{=}$ 9¢

$\boxed{5}\ \boxed{\times}\ \boxed{6}\ \boxed{+}\ \boxed{9}\ \boxed{-}\ \boxed{6}\ \boxed{=}$ 8¢

$\boxed{9}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{+}\ \boxed{6}\ \boxed{=}$ 7¢

N 33

Continue this activity with other numbers such as 333 and 512. Encourage the students to find several solutions using various combinations of the operation keys. Several solutions to each problem are presented here.

333: $\boxed{9}\ \boxed{9}\ \boxed{9}\ \boxed{-}\ \boxed{6}\ \boxed{6}\ \boxed{6}\ \boxed{=}$ 8¢

$\boxed{5}\ \boxed{\times}\ \boxed{6}\ \boxed{6}\ \boxed{+}\ \boxed{9}\ \boxed{-}\ \boxed{6}\ \boxed{=}$ 9¢

$\boxed{6}\ \boxed{5}\ \boxed{\times}\ \boxed{5}\ \boxed{+}\ \boxed{8}\ \boxed{=}$ 7¢

$\boxed{6}\ \boxed{6}\ \boxed{6}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{\div}\ \boxed{6}\ \boxed{=}$ 9¢

512: $\boxed{5}\ \boxed{6}\ \boxed{\times}\ \boxed{9}\ \boxed{+}\ \boxed{8}\ \boxed{=}$ 7¢

$\boxed{5}\ \boxed{6}\ \boxed{5}\ \boxed{-}\ \boxed{5}\ \boxed{9}\ \boxed{+}\ \boxed{6}\ \boxed{=}$ 9¢

$\boxed{5}\ \boxed{9}\ \boxed{8}\ \boxed{-}\ \boxed{8}\ \boxed{6}\ \boxed{=}$ 7¢

$\boxed{8}\ \boxed{\times}\ \boxed{=}\ \boxed{=}$ 4¢

T: Try to put 0.5 on the display of this special calculator.

Let the students have an opportunity to explore this situation. If some need a hint, try asking:

T: What are some fractional names for 0.5?

As the students suggest some names, write them on the board.

$$0.5 = \frac{1}{2} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} = \cdots$$

S: We can divide by 6 or by 8.

S: I used "$\frac{4}{8}$". Press $\boxed{9}\boxed{-}\boxed{5}\boxed{\div}\boxed{8}\boxed{=}$

Other solutions for 0.5 are given below.

0.5: $\boxed{9}\boxed{-}\boxed{6}\boxed{\div}\boxed{6}\boxed{=}$ 6¢

$\boxed{8}\boxed{-}\boxed{6}\boxed{\div}\boxed{=}\boxed{=}$ 6¢

$\boxed{5}\boxed{+}\boxed{=}\boxed{=}\boxed{=}\boxed{\div}\boxed{8}\boxed{\div}\boxed{5}\boxed{=}$ 10¢

$\boxed{9}\boxed{\div}\boxed{6}\boxed{-}\boxed{6}\boxed{+}\boxed{5}\boxed{=}$ 8¢

Continue this activity with other numbers such as 0.25 and 0.1. If some students need a hint, ask them to think of some fractional names; for example, $0.25 = \frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \cdots$ and $0.1 = \frac{1}{10} = \frac{2}{20} = \frac{3}{30} = \cdots$. Try to discuss solutions as soon as they are discovered as such discussion often inspires other students to find new solutions.

0.25: $\boxed{8}\boxed{-}\boxed{6}\boxed{\div}\boxed{8}\boxed{=}$ 6¢

$\boxed{5}\boxed{+}\boxed{5}\boxed{\div}\boxed{8}\boxed{+}\boxed{5}\boxed{-}\boxed{6}\boxed{=}$ 10¢

$\boxed{8}\boxed{-}\boxed{6}\boxed{\div}\boxed{=}\boxed{=}\boxed{=}$ 7¢

N 33

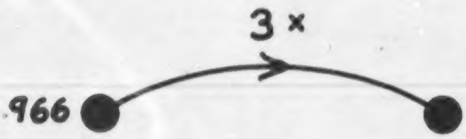
0.1: $\boxed{9}\ \boxed{-}\ \boxed{5}\ \boxed{\div}\ \boxed{5}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$ 8¢

$\boxed{9}\ \boxed{-}\ \boxed{6}\ \boxed{\div}\ \boxed{5}\ \boxed{\div}\ \boxed{6}\ \boxed{=}$ 8¢

$\boxed{5}\ \boxed{+}\ \boxed{5}\ \boxed{\div}\ \boxed{=}\ \boxed{=}$ 6¢

Exercise 2

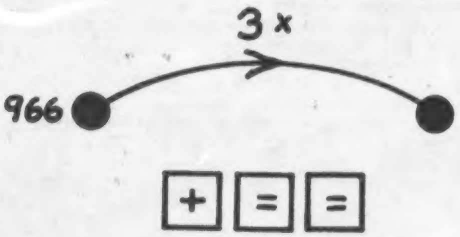
Draw this arrow picture on the board.



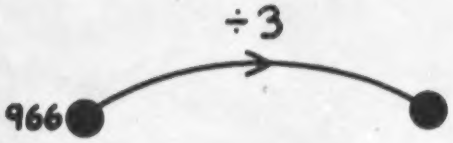
T: Put any number you wish on your calculator. I have chosen 966. Then try to find the cheapest way to multiply your number by 3. Your method should work for any number, not just the one you have chosen.

S: Press $\boxed{+}\boxed{=}\boxed{=}$; that costs three cents.

Add this information to your picture.

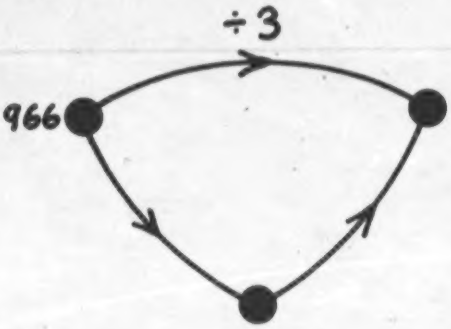


Erase the arrow labels, and label the arrow "÷ 3".



T: Put any number you wish on your hand-calculator. Again I have chosen 966. Try to find the cheapest way to divide your number by 3. Remember we would like a way that works for any number we choose. You might want to use more than one operation.

Add the two arrows shown below to your arrow picture.



T: What is another name for ÷ 3?

S: "$\frac{1}{3}$x".

T: Yes, and what are some other names for $\frac{1}{3}$ which might help us?

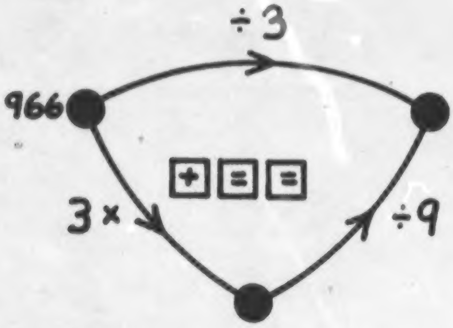
S: "$\frac{3}{9}$".

T: Good. How can we use "$\frac{3}{9}$"?

S: $\frac{3}{9}$x is 3x followed by ÷ 9. We can press $\boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{\div}\ \boxed{9}\ \boxed{=}$.

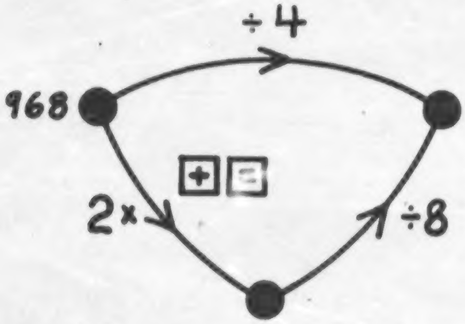
N 33

Add this information to your arrow picture.

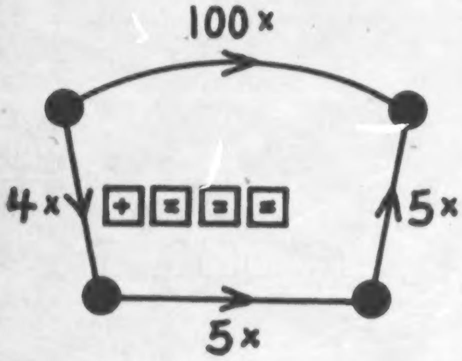


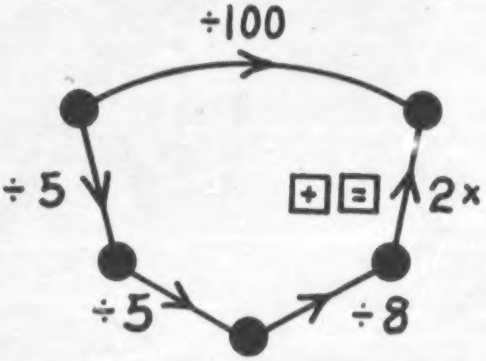
The students should verify that the above method works by pressing the appropriate keys on their hand-calculators.

Repeat this activity for 4× and ÷4. The students should conclude that the cheapest way to multiply any number by 4 is to press $\boxed{+}\boxed{=}\boxed{=}\boxed{=}$. (4¢) Division by 4 may be handled in a manner similar to the above. For example,



If your students like a challenge, ask them to find a way to multiply a number by 100 and to divide a number by 100. Many solutions are possible such as the ones given below.





APPENDIX: Standard Configurations and Standard Mimicomputer Moves

A number can be represented by putting checkers[*] on the squares of the Minicomputer. Each square has a numerical (positional) value, so that the number represented by a checker depends on which square the checker is placed. If several checkers are put on the Minicomputer, the number shown is the sum of the numbers represented by the checkers. A number can be put on the Minicomputer in a variety of ways, but the representation which uses at most one checker on each square and uses checkers to represent a digit 9 or less on each board is usually the easiest to read. In this case, we say that the number is in standard configuration.

Dôing calculations on the Minicomputer involves putting one or more checkers on the various squares and making "plays" to get a specific configuration. In this context, a play is an exchange of checkers in a configuration which does not change the number represented. Some plays are natural to the design of the Minicomputer; we call these standard or usual plays (these are described in detail in this section). There are advantages to establishing such a set of plays:

- They become routine, and a large group of students with varying abilities can follow a calculation without needing to verify every play which is made.
- They lead to standard configurations of numbers.

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[*] In this appendix we refer only to whole numbers and, therefore, use only regular (positive) checkers.

These are the standard Minicomputer moves.

1 + 1 = 2
10 + 10 = 20
100 + 100 = 200
1,000 + 1,000 = 2,000

two checkers on the white square
of a board
=
one checker on the red square
of the same board

2 = 1 + 1
20 = 10 + 10
200 = 100 + 100
2,000 = 1,000 + 1,000

one checker on the red square
of a board
=
two checkers on the white square
of the same board

2 + 2 = 4
20 + 20 = 40
200 + 200 = 400
2,000 + 2,000 = 4,000

two checkers on the red square
of a board
=
one checker on the purple square
of the same board

4 = 2 + 2
40 = 20 + 20
400 = 200 + 200
4,000 = 2,000 + 2,000

one checker on the purple square
of a board
=
two checkers on the red square
of the same board

4 + 4 = 8
40 + 40 = 80
400 + 400 = 800
4,000 + 4,000 = 8,000

two checkers on the purple square
of a board
=
one checker on the brown square
of the same board

8 = 4 + 4
80 = 40 + 40
800 = 400 + 400
8,000 = 4,000 + 4,000

one checker on the brown square
of a board
=
two checkers on the purple square
of the same board

8 + 2 = 10
80 + 20 = 100
800 + 200 = 1,000
8,000 + 2,000 = 10,000

one checker on each of the brown
and red squares of a board
=
one checker on the white square
of the next board to the left

10 = 8 + 2
100 = 80 + 20
1,000 = 800 + 200
10,000 = 8,000 + 2,000

one checker on the white square
of a board
=
one checker on each of the brown
and red squares of the next board
to the right*

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* This provides the link that motivates the introduction of the decimal numbers. In particular, we have that one checker on the white square of the ones' board equals one checker on each of the brown and red squares of a special board (the tenths' board - or as it is often referred to in the lessons, the dimes' board) located to the right of the ones' board and separated from it by a special barrier.